Dynamic band structure measurement in the synthetic space

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Band structure theory plays an essential role in exploring physics in both solid-state systems and photonics. Here, we demonstrate a direct experimental measurement of the dynamic band structure in a synthetic space including the frequency axis of light, realized in a ring resonator under near-resonant dynamic modulation. This synthetic lattice exhibits the physical picture of the evolution of the wave vector reciprocal to the frequency axis in the band structure, analogous to a one-dimensional lattice under an external force. We experimentally measure the trajectories of the dynamic band structure by selectively exciting the band with a continuous wave source with its frequency scanning across the entire energy regime of the band. Our results not only provide a new perspective for exploring the dynamics in fundamental physics of solid-state and photonic systems with the concept of the synthetic dimension but also enable great capability in band structure engineering in photonics.

INTRODUCTION

Band theory, originally developed in solid-state systems for understanding electronic properties of solids (1), has played an important role in the field of photonic crystals to describe the behavior of photons and to manipulate the propagation of light (2) and has been recently applied to explore physics in topological photonics (3, 4). The band structure provides substantial insights into the intrinsic physics of a system. These insights can be further used for band engineering (5–7) including designing bandgaps (8–10) and constructing topological edge states in photonics (11–14).

For a quantum particle in one-dimensional (1D) lattice as described by a tight-binding model with only nearest neighbor interactions, the corresponding band structure is given in (15)

$$\varepsilon_k = 2J \cos(ka)$$

(1)

where $$k$$ is the wave vector, $$a$$ is the lattice constant, and $$J$$ is the interaction strength. For the purpose of this paper, we will refer to this band structure as the stationary band structure. On the other hand, if an external force $$F$$ is applied, the dynamics of the particle in the system can be described by substituting its wave vector components according to (15)

$$k(t) = k - Ft$$

(2)

Since the Brillouin zone is periodic, the wave packet circulates inside the first Brillouin zone as shown in Fig. 1A. The circulation of $$k$$ leads to the oscillation of the probability velocity and causes the wave packet to oscillate in the real space, resulting in an effect known as the Bloch oscillation (16), which has been widely explored both in solid-state and photonic systems (17–25). The combination of Eqs. 1 and 2 gives

$$\varepsilon_k(t) = 2J \cos(ka - Ft)$$

(3)

where $$F = aF$$. Below, we refer to the time-dependent energy band in Eq. 3 as a dynamic band structure. Such a dynamic band structure underlies the physics of Bloch oscillation. However, to the best of our knowledge, there has not been any direct experimental measurement of the band structure evolution over time in $$k$$-space in Eq. 3.

Recently, synthetic dimensions have generated great interest across many areas of photonics (26, 27). The concept of constructing synthetic dimensions is to couple appropriate internal degrees of freedom of physical states to form an extra dimension (28–42). Experimental implementations of synthetic lattices in various photonic platforms have resulted in a number of important experimental demonstrations, such as discrete Talbot effect (36), frequency diffraction and negative refraction (37), synthetic topological photonic insulator (39), and synthetic Hall ladder (41). Similar to standard solid-state systems and photonic structures, all these synthetic lattices are also characterized by stationary band structures. A technique for directly measuring the stationary band structure was demonstrated in (43). Allowing the band structure to vary as a function of time, with Eq. 3 as one of the example, will lead to a much richer set of physics, such as time-reversal operation on light (44–46), super-Bloch oscillations (47, 48), dynamic localization of light under a time-dependent force (49), dynamical classification of topological quantum phases (50), and spectrum control of light (51). To characterize such physics, it is therefore of substantial interest to directly measure dynamic band structure as well.

In this work, we provide a direct experimental measurement of the dynamic band structure associated with a synthetic lattice along the frequency axis of light. We consider a dynamically modulated fiber ring resonator under a near-resonant modulation. Such a cavity supports a synthetic frequency dimension analogous to a 1D lattice under a constant force (24). The physics of such a synthetic lattice can be described by a dynamic band structure in Eq. 3, with $$k_p$$ reciprocal to the frequency axis, changing linearly with the time. We
FIG. 1. Illustration of dynamic band structure. (A) Movement of electrons under an external force \( F \) in \( k \)-space within the first Brillouin zone in a 1D solid-state system. (B) Physical picture of band structure movements for system in (A), which periodically shifts over time in \( k \)-space. (C) Theoretically calculated trajectory of the dynamic band structures based on (B) with \( F' = 0 \), \( F' > 0 \), and \( F' < 0 \) from Eq. 4. (D) A ring resonator dynamically modulated by an EOM. In the near-resonance modulation case, the modulation frequency \( \Omega_m \) is chosen to be slightly different from resonant frequency \( \Omega_R \), while \( \Delta \) is the modulation detuning. (E) The system in (D) can be mapped into a 1D tight-binding model for photons under an effective force along the synthetic frequency dimension.

RESULTS

Theoretical analysis

We start with further considering the dynamic band structure shown in Eq. 3, which corresponds to a 1D tight-binding lattice under a constant force. It shows that the band evolves in the positive (negative) \( k \) direction over time as illustrated in Fig. 1B for \( F > 0 \) (\( F < 0 \)). By exciting the band with a CW source at a linearly dependent frequency, we obtain the relation

\[ \varepsilon_k(t) = \xi t = 2\cos(ka - Ft) \]  

As an illustration shown in Fig. 1B, at any given time \( t_m \) (\( m = 1 \), 2, and 3), the corresponding energy \( \varepsilon_k(t_m) \) excites two points on the band at wave vectors \( k_{m1} \) and \( k_{m2} \) that satisfy Eq. 4. These wave vectors can be determined by analyzing the time-dependent transmittance coefficients of the modulated ring system (43). Therefore, by obtaining \( [k_{m1}, \varepsilon_k(t_m)] \), one can determine the dynamic band structure as shown in Fig. 1C. Different twisted trajectories of the dynamic band structure are found under different external forces, as the force pushes the band moving towards one direction in the \( k \)-space.

We now provide a more detailed calculation on the dynamic band structure, for the dynamically modulated ring resonator structure as shown in Fig. 1D. A ring resonator undergoing dynamic modulations forms a 1D synthetic lattice along the frequency axis of light (34, 52–54). In the absence of the group velocity dispersion, the resonator supports a set of modes with their frequencies equally spaced. If the central resonant frequency is set at \( \omega_n \), the \( n \)th resonant mode has the frequency \( \omega_n = \omega_0 + n\Delta \), where \( n \) is an integer, \( \Omega_R = 2\pi c / n g L \) is the free spectral range (FSR), \( n_g \) is the group index, \( L \) is the length of the ring resonator, and \( c \) is the speed of light. The resonant modes can be coupled together by introducing a time-dependent refractive index modulation, implemented by an electro-optic phase modulator (EOM) with a modulation frequency \( \Omega_m \approx \Omega_R \).

If we consider the weak modulation limit and assume the modulation couples only the nearest neighbor modes, the Hamiltonian of the system is described in (24)

\[ H = \sum_n \omega_n a_n^\dagger a_n + 2g \cos(\Omega_m t) \sum_n (a_n^\dagger a_{n+1} + a_n a_{n+1}) \]  

where \( a_n^\dagger (a_n) \) is the creation (annihilation) operator for the \( n \)th mode, and \( g \) is the modulation strength. Here, we consider the near-resonance modulation case, where \( \Delta = \Omega_m - \Omega_m \ll \Omega_R \) represents the detuning between the modulation frequency and the FSR. The Hamiltonian in Eq. 5 can be simplified by going to the interaction picture and taking the rotating-wave approximation that results in (55)

\[ H_{\text{c}} = g \sum_n (c_{n+1}^\dagger c_n e^{i\Delta t} + c_n^\dagger c_{n+1} e^{-i\Delta t}) \]  

with \( a_n = c_n e^{i\omega_n t} \). In the 1D lattice case, a photon under a constant effective force can be described by a time-independent Hamiltonian

\[ H_{\text{c}} = g \sum_n (c_{n+1}^\dagger c_n + c_n^\dagger c_{n+1}) + \sum_n n \Delta c_n^\dagger c_n \]  

which is equivalent to the Hamiltonian in Eq. 6 by a gauge transformation \( | \Psi \rangle = \sum_n v_n c_n^\dagger | 0 \rangle \rightarrow | \Psi \rangle = \sum_n v_n e^{-i\Delta t} c_n^\dagger | 0 \rangle \) (24). Hence, the strength of the effective force for photons generated in the modulated ring cavity is \( F = | \Delta / \Omega_R | \) at the synthetic frequency dimension as shown in Fig. 1E.

The Hamiltonian in Eq. 6 can be converted into the \( k_f \)-space, where \( k_f \) is the wave vector reciprocal to the frequency dimension

\[ H_{k_f} = \sum_{k_f} 2g c_{k_f}^\dagger c_{k_f} \cos(k_f \Omega_R - \Delta t) \]  

It indicates that the system has an instantaneous photonic band structure

\[ \varepsilon_{k_f}(t) = 2g \cos(k_f \Omega_R - \Delta t) \]  

within the first Brillouin zone \( k_f \in [-\pi / \Omega_R, \pi / \Omega_R] \).

The time-resolved transmission spectroscopy in (43) is also applicable to the near-resonance modulation case in this work. Suppose the input frequency detuning of the laser source is \( \omega(t) = 2\pi n t \), where \( n \) indicates the input frequency detuning rate. Combining the energy band in \( k_f \)-space in Eq. 9, we can obtain the trajectory of the dynamic band structure for photons under an effective force in the 1D synthetic dimension.
Equation 10 is mathematically equivalent to Eq. 4, which describes the energy band that evolves over time in solid-state system, with \( \epsilon_n(t) = \omega(t) \), \( k = k_f \), and \( F = \Delta \), highlighting the analogy between these two systems. Therefore, this near-resonance modulated ring cavity is expected to exhibit the capability for measuring trajectory of the dynamic band structure and simulate the behavior of the dynamic band structure of electrons under external forces.

In simulations, we consider that the photon state is \( |\Psi\rangle = \sum_n v_n e^{i n\omega t} |0\rangle \) with \( v_n \) being the amplitude of the \( n \)th resonant mode. By applying the Schrödinger equation \( i |\Psi\rangle = H \dot{|\Psi\rangle} \) with \( H \) defined in Eq. 6, where the eigenfrequencies of modes have been eliminated in the interaction picture, we obtain the coupled-mode equation for the \( n \)th mode

\[
\frac{d}{dt} v_n(t) = -g(v_{n+1} e^{-i\Delta t} + v_{n-1} e^{i\Delta t}) - \gamma v_n + i \sqrt{\gamma} S_m(t) \delta_{n0} \tag{11}
\]

where \( \gamma \) is the total loss, \( \gamma_c \) is the waveguide-cavity coupling strength, \( S_m(t) \) is the input laser source at the frequency \( \omega_0 \), and \( S_{out,n}(t) \) is the output field in the drop port at \( \omega_n \). The drop-port transmission of the field gives

\[
T_{out} = \left| \sum_n S_{out,n}(t) e^{-i\omega_0 t} \right|^2 = \gamma_c \left| \sum_n v_n(t) e^{-i\omega_0 t} \right|^2 \tag{12}
\]

In the on-resonance modulation case with \( \Delta = 0 \), Dutt et al. (43) have demonstrated that a stationary band structure can be read out directly by time-resolved transmission spectroscopy. One can obtain the drop-port transmission signal in Eq. 13 by scanning the input laser frequency \( \omega(t) \) linearly over time. The transmission signal can be broken into time slices with a fixed time window that equals to one roundtrip time \( 2\pi/\Omega_R \), and each time slice gives the detection of a band at the energy \( \omega \) and the wave vector \( k_f \) (the fast time variable within this time slice) (43). Stacking up these time slices along the input frequency detuning axis yields the time-resolved transmission, which directly reveals the band structure of the system.

### Experimental results

We experimentally demonstrate the measurement of the dynamic band structure by using a fiber ring resonator as shown in Fig. 2 (see Materials and Methods). We first calibrate the scan time \( t \) on the oscilloscope and scanning frequency axis \( \omega(t) \) to convert the slow time axis into the frequency axis, as well as to calculate the input frequency detuning rate \( \eta \). We can apply a triangular ramp signal to the frequency modulation input of laser to finely scan the laser’s frequency, while the central wavelength locates at 1540.56 nm. The length of the ring is \( L \sim 9.3 \text{ m} \), corresponding to an FSR of \( \Omega_R = 2\pi \cdot 22 \text{ MHz} \) in theory. When the applied ramp signal is 100 Hz with an amplitude of 0.5 V (corresponding to \( \sim 5 \text{ GHz} \) scanning range), the time duration for the laser wavelength to scan between adjacent resonant modes is 22 \( \mu \text{s} \). Since one roundtrip time is 45.5 ns, we obtain the input frequency detuning rate \( \eta = 0.002 \). In addition, in the absence of modulation, the measured full width at half maximum of the resonant mode is 1.56 MHz, corresponding to a quality factor of \( Q \approx 1.25 \times 10^8 \). To construct a 1D synthetic frequency lattice, we apply a sinusoidal radio frequency (RF) signal with a form of \( V_M(t) = \sqrt{2} V \cos(\Omega_M t) \) to the EOM. Moreover, accurate FSR is needed as we shall explore the trajectories of band evolutions under different effective forces. We therefore vary the modulation frequency of the EOM till the modulation sidebands are being fully overlapped with the adjacent resonant modes. It gives \( \Omega_M = 2\pi \cdot 22 \text{ MHz} \), in agreement with the theoretical value.

We measure the trajectories of the dynamic band structure while changing the modulation frequency \( \Omega_M \) slightly from \( \Omega_R \) as plotted in Fig. 3 (A to E) by using the time-resolved transmission spectroscopy (43). As for a fixed coupling strength \( V = 1.5 \text{ V} \), the system exhibits stationary band structure when \( \Omega_M = \Omega_R \) (Fig. 3D), corresponding to the on-resonance modulation case without the effective force. Once the modulation is off-resonant, the band structure starts to evolve in time due to the effective force, and we see the trajectories of such evolutions. Moreover, the trajectory evolution of the band structure over time is toward the opposite \( k_f \) direction when modulation detunings are opposite as shown in Fig. 3 (A and E). The splitting trajectory of the band structure is observed when the modulation detuning is increased, while the energy (frequency) window remains as \( \sim 3.84 \text{ MHz} \). It means that photons oscillate faster within one energy band when the effective force becomes stronger. The experimental results show excellent agreement with the theoretically

![Fig. 2. Experimental setup. The laser’s frequency is finely scanned by applying an external linear voltage ramp signal to its frequency modulation input. PC, polarization controller; DWDM, dense wavelength division multiplexing; PD, photodiode.](http://advances.sciencemag.org/)

calculated trajectories of the band structure in 1D solid-state system (Fig. 3, F to J) based on Eq. 4 and numerically simulated trajectories of the band structure for photons under the near-resonance ring model (Fig. 3, K to O) based on Eqs. 11 to 13. The parameters used in both models are calculated on the basis of the experimental conditions. The widths of bands in Fig. 3 are mainly due to the waveguide-cavity coupling $\gamma$, if the system is near the critical coupling regime. A reduced $\gamma$, by using higher-ratio optical couplers combined with tuning the amplifier inside the ring to reach the critical coupling may potentially narrow down the bands, as long as the output signal intensity is still above the noise level of the system. The method is also capable of measuring trajectories of the dynamic band structure with long-range coupling. In the Supplementary Materials (Fig. S1), we show the experimentally measured band structures with only the long-range coupling near either twice or thrice the FSR, which agrees well with both theoretically calculated and numerically simulated trajectories of the band structure. Such twice or thrice FSR modulation introduces only next-nearest- or next-next-nearest-neighbor couplings between resonant modes. It ends up to two or three independent synthetic lattices, where the first Brillouin zones have been halved and trisected. Further opportunity of exploring the synthetic lattice with the long-range couplings is possible by applying an external modulation signal including both the near-resonance modulation frequency and high-order modulation frequencies (43).

We then study the effect of the modulation strength on the trajectory of the band structure and transmission spectrum, which are shown in Fig. 4. Trajectories of dynamic band structures in Fig. 4 (A to D) are obtained by fixing the modulation frequency as 20 MHz and varying the applied RF signal amplitude $V$, which shows good matching with numerically simulated results in Fig. 4 (E to H). Typical Lorentzian resonance of the unmodulated cavity is seen for $V = 0$. The splitting in trajectories of the band structure becomes more obvious when $V$ is increased. Under large modulation detuning and strength, distinct sidebands begin to show up, which can be seen in the transmission spectra in Fig. 4. The increase of the energy window of the band versus increasing the modulation amplitude is also observed, which is consistent to Eq. 10. The low signal-to-noise ratio of the measured transmission spectra in Fig. 4A is due to the small record time of the oscilloscope to read out the dynamic band structure with high accuracy (see fig. S2). The transmission spectrum in fig. S2A with $\omega(t) = 11\Omega_R$ shows high signal-to-noise ratio. As the record range becomes smaller with $\omega(t) = 1.8\Omega_R$, the spectra become noisy as shown in fig. S2B. The roughness of the transmission spectra in Fig. 4 (B to D) originates from the small displayed frequency detuning range, which contains about 140 sinusoidal signal periods in every transmission spectrum (see fig. S3). Further improvement in the noise lever is possible if we can miniaturize our design into

**Fig. 3.** Trajectories of the dynamic band structure under near-resonance modulation with fixed modulation strength. (A to E) Experimentally measured band structures with modulation frequency $\Omega_M = 21.0, 21.5, 21.8, 22.0,$ and $23.0$ MHz, respectively, and a fixed applied RF voltage of $V = 1.5$ V. (F to J) Theoretically calculated trajectories of the band structure in 1D solid-state system under external force $F = -0.285, 0.143, 0.057, 0,$ and $-0.285$, respectively, and a fixed interaction strength $J = 0.2$ based on Eq. 4. (K to O) Numerically simulated trajectories of the band structure with modulation detuning $\Delta = 0.285, 0.143, 0.057, 0,$ and $-0.285$, respectively, and a fixed coupling strength $g = 0.2$ and $\gamma = 0.2$ based on Eqs. 11 to 13. The bottom x axis in (A) to (E) and (K) to (O) represents one roundtrip time with period of $2\pi/\Omega_R$, while x axis in (F) to (J) presents the first Brillouin zone in solids with period of $2\pi/a$. The y axis represents the frequency detuning of the input laser from the resonant frequency normalized to FSR. The sign of modulation detuning is mapped to the sign of the effective force and hence controls the band shapes.

**Fig. 4.** Trajectories of dynamic band structures and transmission spectra with fixed modulation detuning. (A to D) Experimentally observed trajectories of the dynamic band structure with fixed modulation frequency $\Omega_M = 20$ MHz and measured transmission spectra from drop port. (E to H) Numerically simulated trajectories of the dynamic band structure with fixed modulation detuning $\Delta = 0.57$ and calculated transmission spectra from Eq. 13. a.u., arbitrary units.
chip scale. Our experimental measurements indicate that we observe the trajectory of the dynamic band structure along the \( k_f \)-space (see Eq. 10), and our experiment therefore simulates the trajectory of a moving momentum associated in the 1D solid-state system under a constant force (see Eq. 4).

**DISCUSSION**

The approach of dynamic band structure measurement we have experimentally proposed here is suitable for several possibilities and applications in quantum simulation and quantum information processing (56). One possibility is to study various band structures that correspond to different coupling configurations in the tight-binding lattice, such as a lattice including long-range coupling and a lattice under a time-dependent force, by changing the modulation pattern. The 200-MHz bandwidth of the arbitrary waveform generator (AWG) that we used in our setup can support ninth-order long-range coupling. The range of coupling can be further extended by using an AWG with higher bandwidth or using lager rings with a smaller FSR. If we further introduce a complex external voltage with a time-dependent modulation frequency, it is possible to explore the band structure dynamics under time-dependent force, such as in the regime of dynamic localization (49). On the other hand, the dynamic band structure can also be extended beyond 1D synthetic space by using real-space dimensions (34, 35), additional frequency dimensions (29), or other synthetic dimensions such as pseudospin (41) and orbital angular momentum (32, 38). Other interesting applications include studies of dynamic physics with interacting Hamiltonians by introducing optical nonlinearities (57) and/or in the quantum regime (58). Moreover, the recent advent of the on-chip lithium niobate microring platform shows promise for measuring dynamic band structure in synthetic dimensions in integrated photonics (53, 54). We note that the bandwidth of commercial measurement equipment can be up to 40 GHz, then the corresponding radius will reduce to 600 \( \mu \)m for lithium niobate microring resonators. Further miniaturization of the structure can be accomplished with the availability of higher bandwidth measurement equipment.

In summary, we provide a direct experimental measurement of the dynamic band structure associated with a synthetic frequency lattice, constructed by a dynamically modulated ring cavity. The movement of the wave vector along the synthetic momentum reciprocal to the frequency dimension is observed by the time-resolved band structure spectroscopy. Experimental measurements and numerical simulations show great agreements with the calculated trajectories of the dynamic band structure from the solid-state theory. Various dynamic band shapes are observed, which gives the evidence of directly observing the dynamic band structure. Our work hence opens a different perspective on studying the dynamics of the band structure for understanding the behavior of electrons or photons in either solid-state or photonic systems.

**MATERIALS AND METHODS**

**Experimental setup**

The EOM is a lithium niobate EOM with a 10-GHz bandwidth. We excite the ring with a laser having a narrow linewidth of 200 kHz and a wide tunable range from 1520 to 1570 nm. The laser’s frequency can be finely scanned over 30-GHz range (\( \sim 0.24 \) nm in wavelength). A polarization controller is used to align the polarization of laser circulating in the ring to the principle axis of EOM. The first fiber coupler couples 0.5% of the laser source to the ring cavity, while a drop port is incorporated using a second 99.5:0.5 fiber coupler. A semiconductor optical amplifier (SOA) with a maximum gain of 10 dB is used to compensate various losses in the cavity to achieve a high quality factor (Q) for the cavity. The amplified spontaneous emission noise from SOA is filtered by using a dense wavelength division multiplexing with a central wavelength of 1540.56 nm (international telecommunication union channel 46) and a bandwidth of 100 GHz, which is large enough to allow required resonant modes experiencing flat transmission. The through port and drop port outputs are detected by fast InGaAs photodiodes (850 to 1650 nm width of 100 GHz, which is large enough to allow required resonant modes experiencing flat transmission. The through port and drop port outputs are detected by fast InGaAs photodiodes (850 to 1650 nm)

**REFERENCES AND NOTES**
