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Supplementary Materials for

Emergent hydrodynamic bound states between magnetically powered micropropellers

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Published 26 January 2018, *Sci. Adv.* **4**, eaap9379 (2018)
DOI: 10.1126/sciadv.aap9379

The PDF file includes:

- Theoretical Model

Other Supplementary Material for this manuscript includes the following:
(available at advances.sciencemag.org/cgi/content/full/4/1/eaap9379/DC1)

- movie S1 (.avi format). Movie corresponding to Fig. 1E.
- movie S2 (.avi format). Movie corresponding to Fig. 2B.
- movie S3 (.avi format). Movie corresponding to Fig. 3A.
- movie S4 (.avi format). Movie corresponding to Fig. 4A.

THEORETICAL MODEL

At any point of the space, the flow created by a sphere of radius a, rotating with angular velocity Ω at a distance h from a solid wall can be derived from the Blake's tensor, see Eq.1 in the main text.
Let us consider two rotors, with radii a_1 and a_2, located at positions $\vec{r}_1 = (x_1, y_1, z_1)$ and $\vec{r}_2 = (x_2, y_2, z_2)$, elevations from the surface h_1 and h_2, and rotating with angular velocities $\vec{\Omega}_1 = (0, \Omega_1, 0)$ and $\vec{\Omega}_2 = (0, \Omega_2, 0)$ respectively. The flow at any point of the space will be the direct sum of that created by both particles. In the asymptotic limit in which we treat them as point particles, the hydrodynamic velocity they create can be taken as the velocity of the flux at the particle center. We should note that this approximation leads to a singularity when computing the term corresponding to the flow created by the particle itself, requiring for a more accurate treatment that considers the finite size of the rotors. Using this procedure, we find that the hydrodynamic velocities of the two particles are given by

$$\begin{cases} v_{x,1} = \dfrac{\Omega_1 a_1^5}{8h_1^4} + \Omega_2 a_2^3 \left(6h_1 \dfrac{(x_1 - x_2)^2}{(\Delta^2 + (h_1 + h_2)^2)^{5/2}} + \dfrac{h_1 - h_2}{(\Delta^2 + (h_1 - h_2)^2)^{3/2}} + \dfrac{h_2 - h_1}{(\Delta^2 + (h_1 + h_2)^2)^{3/2}} \right) \\ v_{x,2} = \dfrac{\Omega_2 a_2^5}{8h_2^4} + \Omega_1 a_1^3 \left(6h_2 \dfrac{(x_1 - x_2)^2}{(\Delta^2 + (h_1 + h_2)^2)^{5/2}} + \dfrac{h_2 - h_1}{(\Delta^2 + (h_1 - h_2)^2)^{3/2}} + \dfrac{h_1 - h_2}{(\Delta^2 + (h_1 + h_2)^2)^{3/2}} \right) \\ v_{y,1} = 6h_1 \Omega_2 a_2^3 \dfrac{(x_2 - x_1)(y_2 - y_1)}{(\Delta^2 + (h_1 + h_2)^2)^{5/2}} \\ v_{y,2} = 6h_2 \Omega_1 a_1^3 \dfrac{(x_2 - x_1)(y_2 - y_1)}{(\Delta^2 + (h_1 + h_2)^2)^{5/2}} \\ v_{z,1} = \Omega_2 a_2^3 \left(\dfrac{x_2 - x_1}{(\Delta^2 + (h_1 - h_2)^2)^{3/2}} + \dfrac{x_1 - x_2}{(\Delta^2 + (h_1 + h_2)^2)^{3/2}} + 6h_1 \dfrac{(x_1 - x_2)(h_1 + h_2)}{(\Delta^2 + (h_1 + h_2)^2)^{5/2}} \right) \\ v_{z,2} = \Omega_1 a_1^3 \left(\dfrac{x_1 - x_2}{(\Delta^2 + (h_1 - h_2)^2)^{3/2}} + \dfrac{x_2 - x_1}{(\Delta^2 + (h_1 + h_2)^2)^{3/2}} + 6h_2 \dfrac{(x_2 - x_1)(h_1 + h_2)}{(\Delta^2 + (h_1 + h_2)^2)^{5/2}} \right) \end{cases} \tag{1}$$

with $\Delta^2 \equiv (x_2 - x_1)^2 + (y_2 - y_1)^2$. We can express the velocities of each particle in terms of the center of velocity of the pair, v_{cv} and the relative velocity v_{rel}, as

$$\begin{cases} \vec{x}_{cv} \equiv \dfrac{\vec{x}_1 + \vec{x}_2}{2} \\ \vec{x}_{rel} \equiv \dfrac{\vec{x}_2 - \vec{x}_1}{2} \\ \vec{v}_{cv} \equiv \dfrac{\vec{v}_1 + \vec{v}_2}{2} \\ \vec{v}_{rel} \equiv \dfrac{\vec{v}_2 - \vec{v}_1}{2} \end{cases} \tag{2}$$

We note that here, for compactness. we have defined $\vec{v}_{rel} \equiv \vec{v}_2 - \vec{v}_1/2$, however the data and the fits in the main text were done for $\vec{v}_{rel} \equiv \vec{v}_2 - \vec{v}_1$. With this system of coordinates, the components of the velocity in Eqs.(1) become

$$\begin{cases}
v_{x,cv} = \dfrac{\Omega_1 a_1^5}{16(h_{cv}-h_{rel})^4} + \dfrac{\Omega_2 a_2^5}{16(h_{cv}+h_{rel})^4} + \dfrac{\Omega_2 a_2^3}{8}\left(\dfrac{3(h_{cv}-h_{rel})x_{rel}^2}{(x_{rel}^2+y_{rel}^2+h_{cv}^2)^{5/2}} - \dfrac{h_{rel}}{(x_{rel}^2+y_{rel}^2+h_{rel}^2)^{3/2}} + \right. \\
\quad \left.\dfrac{h_{rel}}{(x_{rel}^2+y_{rel}^2+h_{cv}^2)^{3/2}}\right) + \dfrac{\Omega_1 a_1^3}{8}\left(\dfrac{3(h_{cv}+h_{rel})x_{rel}^2}{(x_{rel}^2+y_{rel}^2+h_{cv}^2)^{5/2}} + \dfrac{h_{rel}}{(x_{rel}^2+y_{rel}^2+h_{rel}^2)^{3/2}} - \right. \\
\quad \left.\dfrac{h_{rel}}{(x_{rel}^2+y_{rel}^2+h_{cv}^2)^{3/2}}\right) \\
v_{x,rel} = \dfrac{\Omega_2 a_2^5}{16(h_{cv}+h_{rel})^4} - \dfrac{\Omega_1 a_1^5}{16(h_{cv}-h_{rel})^4} + \dfrac{\Omega_1 a_1^3}{8}\left(\dfrac{3(h_{cv}+h_{rel})x_{rel}^2}{(x_{rel}^2+y_{rel}^2+h_{cv}^2)^{5/2}} + \dfrac{h_{rel}}{(x_{rel}^2+y_{rel}^2+h_{rel}^2)^{3/2}} - \right. \\
\quad \left.\dfrac{h_{rel}}{(x_{rel}^2+y_{rel}^2+h_{cv}^2)^{3/2}}\right) - \dfrac{\Omega_2 a_2^3}{8}\left(\dfrac{3(h_{cv}-h_{rel})x_{rel}^2}{(x_{rel}^2+y_{rel}^2+h_{cv}^2)^{5/2}} - \dfrac{h_{rel}}{(x_{rel}^2+y_{rel}^2+h_{rel}^2)^{3/2}} + \right. \\
\quad \left.\dfrac{h_{rel}}{(x_{rel}^2+y_{rel}^2+h_{cv}^2)^{3/2}}\right) \\
v_{y,cv} = \dfrac{3}{8}\left(\dfrac{\Omega_2 a_2^3(h_{cv}-h_{rel})x_{rel}y_{rel}}{(x_{rel}^2+y_{rel}^2+h_{cv}^2)^{5/2}} + \dfrac{\Omega_1 a_1^3(h_{cv}+h_{rel})x_{rel}y_{rel}}{(x_{rel}^2+y_{rel}^2+h_{cv}^2)^{5/2}}\right) \\
v_{y,rel} = \dfrac{3}{8}\left(\dfrac{\Omega_1 a_1^3(h_{cv}+h_{rel})x_{rel}y_{rel}}{(x_{rel}^2+y_{rel}^2+h_{cv}^2)^{5/2}} - \dfrac{\Omega_2 a_2^3(h_{cv}-h_{rel})x_{rel}y_{rel}}{(x_{rel}^2+y_{rel}^2+h_{cv}^2)^{5/2}}\right) \\
v_{z,cv} = \dfrac{\Omega_1 a_1^3}{8}\left(\dfrac{x_{rel}}{(x_{rel}^2+y_{rel}^2+h_{cv}^2)^{3/2}} - \dfrac{x_{rel}}{(x_{rel}^2+y_{rel}^2+h_{rel}^2)^{3/2}} + \dfrac{3(h_{cv}+h_{rel})x_{rel}h_{cv}}{(x_{rel}^2+y_{rel}^2+h_{cv}^2)^{5/2}}\right) + \\
\quad \dfrac{\Omega_2 a_2^3}{8}\left(\dfrac{x_{rel}}{(x_{rel}^2+y_{rel}^2+h_{rel}^2)^{3/2}} - \dfrac{x_{rel}}{(x_{rel}^2+y_{rel}^2+h_{cv}^2)^{3/2}} - \dfrac{3(h_{cv}-h_{rel})x_{rel}h_{cv}}{(x_{rel}^2+y_{rel}^2+h_{cv}^2)^{5/2}}\right) \\
v_{z,rel} = \dfrac{\Omega_1 a_1^3}{8}\left(\dfrac{x_{rel}}{(x_{rel}^2+y_{rel}^2+h_{cv}^2)^{3/2}} - \dfrac{x_{rel}}{(x_{rel}^2+y_{rel}^2+h_{rel}^2)^{3/2}} + \dfrac{3(h_{cv}+h_{rel})x_{rel}h_{cv}}{(x_{rel}^2+y_{rel}^2+h_{cv}^2)^{5/2}}\right) + \\
\quad \dfrac{\Omega_2 a_2^3}{8}\left(\dfrac{x_{rel}}{(x_{rel}^2+y_{rel}^2+h_{cv}^2)^{3/2}} - \dfrac{x_{rel}}{(x_{rel}^2+y_{rel}^2+h_{rel}^2)^{3/2}} + \dfrac{3(h_{cv}-h_{rel})x_{rel}h_{cv}}{(x_{rel}^2+y_{rel}^2+h_{cv}^2)^{5/2}}\right)
\end{cases} \tag{3}$$

Effect of gravity

We consider the effect of the vertical forces as a stokeslet and add its contribution to Eqs.3 as

$$
\begin{cases}
u_{x,cv} = & \frac{K(h_0-h_{cv}+h_{rel})}{32\pi\eta}\left[\frac{x_{rel}h_{rel}}{(x_{rel}^2+y_{rel}^2+h_{rel}^2)^{3/2}}+\frac{x_{rel}(h_{cv}-2h_{rel})}{(x_{rel}^2+y_{rel}^2+h_{cv}^2)^{3/2}}\right. \\
& \left.-\frac{3x_{rel}h_{cv}(h_{cv}+h_{rel})(h_{cv}-h_{rel})}{(x_{rel}^2+y_{rel}^2+h_{cv}^2)^{5/2}}\right]+\frac{K(h_0-h_{cv}-h_{rel})}{32\pi\eta} \\
& \left[\frac{x_{rel}h_{rel}}{(x_{rel}^2+y_{rel}^2+h_{rel}^2)^{3/2}}-\frac{x_{rel}(2h_{rel}-h_{cv})}{(x_{rel}^2+y_{rel}^2+h_{cv}^2)^{3/2}}+\frac{3x_{rel}h_{cv}(h_{cv}+h_{rel})(h_{cv}-h_{rel})}{(x_{rel}^2+y_{rel}^2+h_{cv}^2)^{3/2}}\right] \\
u_{x,rel} = & \frac{K(h_0-h_{cv}+h_{rel})}{32\pi\eta}\left[\frac{x_{rel}h_{rel}}{(x_{rel}^2+y_{rel}^2+h_{rel}^2)^{3/2}}+\frac{x_{rel}(h_{cv}-2h_{rel})}{(x_{rel}^2+y_{rel}^2+h_{cv}^2)^{3/2}}\right. \\
& \left.-\frac{3x_{rel}h_{cv}(h_{cv}+h_{rel})(h_{cv}-h_{rel})}{(x_{rel}^2+y_{rel}^2+h_{cv}^2)^{5/2}}\right]-\frac{K(h_0-h_{cv}-h_{rel})}{32\pi\eta} \\
& \left[\frac{x_{rel}h_{rel}}{(x_{rel}^2+y_{rel}^2+h_{rel}^2)^{3/2}}-\frac{x_{rel}(2h_{rel}-h_{cv})}{(x_{rel}^2+y_{rel}^2+h_{cv}^2)^{3/2}}+\frac{3x_{rel}h_{cv}(h_{cv}+h_{rel})(h_{cv}-h_{rel})}{(x_{rel}^2+y_{rel}^2+h_{cv}^2)^{3/2}}\right] \\
u_{y,cv} = & \frac{K(h_0-h_{cv}+h_{rel})}{32\pi\eta}\left[\frac{y_{rel}h_{rel}}{(x_{rel}^2+y_{rel}^2+h_{rel}^2)^{3/2}}+\frac{y_{rel}(h_{cv}-2h_{rel})}{(x_{rel}^2+y_{rel}^2+h_{cv}^2)^{3/2}}\right. \\
& \left.-\frac{3y_{rel}h_{cv}(h_{cv}+h_{rel})(h_{cv}-h_{rel})}{(x_{rel}^2+y_{rel}^2+h_{cv}^2)^{5/2}}\right]+\frac{K(h_0-h_{cv}-h_{rel})}{32\pi\eta} \\
& \left[\frac{y_{rel}h_{rel}}{(x_{rel}^2+y_{rel}^2+h_{rel}^2)^{3/2}}-\frac{y_{rel}(2h_{rel}-h_{cv})}{(x_{rel}^2+y_{rel}^2+h_{cv}^2)^{3/2}}+\frac{3y_{rel}h_{cv}(h_{cv}+h_{rel})(h_{cv}-h_{rel})}{(x_{rel}^2+y_{rel}^2+h_{cv}^2)^{3/2}}\right] \\
u_{y,rel} = & \frac{K(h_0-h_{cv}+h_{rel})}{32\pi\eta}\left[\frac{y_{rel}h_{rel}}{(x_{rel}^2+y_{rel}^2+h_{rel}^2)^{3/2}}+\frac{y_{rel}(h_{cv}-2h_{rel})}{(x_{rel}^2+y_{rel}^2+h_{cv}^2)^{3/2}}\right. \\
& \left.-\frac{3y_{rel}h_{cv}(h_{cv}+h_{rel})(h_{cv}-h_{rel})}{(x_{rel}^2+y_{rel}^2+h_{cv}^2)^{5/2}}\right]-\frac{K(h_0-h_{cv}-h_{rel})}{32\pi\eta} \\
& \left[\frac{y_{rel}h_{rel}}{(x_{rel}^2+y_{rel}^2+h_{rel}^2)^{3/2}}-\frac{y_{rel}(2h_{rel}-h_{cv})}{(x_{rel}^2+y_{rel}^2+h_{cv}^2)^{3/2}}+\frac{3y_{rel}h_{cv}(h_{cv}+h_{rel})(h_{cv}-h_{rel})}{(x_{rel}^2+y_{rel}^2+h_{cv}^2)^{3/2}}\right] \\
u_{z,cv} = & \frac{K(h_0-h_{cv})}{16\pi\eta}\left[\frac{1}{(x_{rel}^2+y_{rel}^2+h_{rel}^2)^{1/2}}+\frac{h_{rel}^2}{(x_{rel}^2+y_{rel}^2+h_{rel}^2)^{3/2}}-\frac{1}{(x_{rel}^2+y_{rel}^2+h_{cv}^2)^{1/2}}-\right. \\
& \left.\frac{h_{rel}^2}{(x_{rel}^2+y_{rel}^2+h_{cv}^2)^{3/2}}\right]+\frac{K(h_0-h_{cv}-h_{rel})}{96\pi\eta a_2(h_{cv}+h_{rel})^3}\left[16(h_{cv}+h_{rel})^3+7a_2^3-24a_2(h_{cv}+h_{rel})^2\right] \\
& +\frac{K(h_0-h_{cv}+h_{rel})}{96\pi\eta a_1(h_{cv}-h_{rel})^3}\left[16(h_{cv}-h_{rel})^3+7a_1^3-24a_1(h_{cv}-h_{rel})^2\right] \\
u_{z,rel} = & \frac{Kh_{rel}}{16\pi\eta}\left[\frac{1}{(x_{rel}^2+y_{rel}^2+h_{rel}^2)^{1/2}}+\frac{h_{rel}^2}{(x_{rel}^2+y_{rel}^2+h_{rel}^2)^{3/2}}-\frac{1}{(x_{rel}^2+y_{rel}^2+h_{cv}^2)^{1/2}}-\right. \\
& \left.\frac{h_{rel}^2}{(x_{rel}^2+y_{rel}^2+h_{cv}^2)^{3/2}}\right]+\frac{K(h_0-h_{cv}-h_{rel})}{96\pi\eta a_2(h_{cv}+h_{rel})^3}\left[16(h_{cv}+h_{rel})^3+7a_2^3-24a_2(h_{cv}+h_{rel})^2\right] \\
& -\frac{K(h_0-h_{cv}+h_{rel})}{96\pi\eta a_1(h_{cv}-h_{rel})^3}\left[16(h_{cv}-h_{rel})^3+7a_1^3-24a_1(h_{cv}-h_{rel})^2\right]
\end{cases}
\tag{4}
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