Supplementary Materials for

Intermittent inverse-square Lévy walks are optimal for finding targets of all sizes

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A Preliminary theoretical results

For general definitions regarding the model, see Methods in the main text. Let us, however, recall here few definitions that will be used extensively.

The torus $T_n$ is identified with the set $[-\sqrt{n}/2, \sqrt{n}/2]^2$ in the infinite plain $\mathbb{R}^2$. Consider $\mu \in (1,3]$ and maximal step $\ell_{\max} > 1$ (possibly $\ell_{\max} = \infty$). A Lévy walk $Z^\mu$ on $\mathbb{R}^2$ (or $X^\mu$ on $T_n$), with maximal step $\ell_{\max} > 1$, is the random walk process whose step-lengths are distributed according to

$$p(\ell) = \begin{cases} a & \text{if } \ell \leq 1 \\ a\ell^{-\mu} & \text{if } \ell \in (1, \ell_{\max}) \\ 0 & \text{if } \ell \geq \ell_{\max} \end{cases},$$

(1)

where $a = (1 + \int_1^{\ell_{\max}} \ell^{-\mu}d\ell)^{-1}$ is the normalization factor. When considering a Lévy process on the torus, we shall take $\ell_{\max} = \sqrt{n}/2$. Recall also, that when $\mu = 2$, we refer to the process as a Cauchy walk. The Cauchy walk on the torus is denoted $X^{\text{cauchy}}$.

In addition, we shall extensively use the following definition.

**Definition 1.** Given a target $S$, the extended set $B(S)$ is the set of nodes at distance at most 1 from $S$. Note that since the radius of detection is 1, the searcher detects $S$ if and only if it is located in $B(S)$.

A.1 Expectations and variances of step-lengths

**Claim 2.** Consider the Lévy walk $Z^\mu$ (or $X^\mu$) with maximal step length $\ell_{\max}$. The average length of a step (and hence the average time to take a step) is

$$\tau = \begin{cases} \Theta(\ell_{\max}^{2-\mu}) & \text{if } \mu \in (1,2) \\ \Theta(\log \ell_{\max}) & \text{if } \mu = 2 \\ \Theta(1) & \text{if } \mu \in (2,3] \end{cases},$$

(2)

and the variance $\sigma^2$ and second moment $M$ of a step-length are

$$\sigma^2 = \Theta(M) = \begin{cases} \Theta(\ell_{\max}^{3-\mu}) & \text{if } \mu \in (1,3) \\ \Theta(\log \ell_{\max}) & \text{if } \mu = 3 \end{cases}.$$

(3)
Proof. Given the definition of $p$, the expected step-length is

$$\tau = \int_0^1 a\ell \, d\ell + \int_1^{\ell_{\text{max}}} a\ell^{1-\mu} \, d\ell.$$  

The first term is $\frac{a}{2}$, a constant, the second term is $\Theta(\ell_{\text{max}}^{2-\mu})$ if $\mu \neq 2$, and $\Theta(\log \ell_{\text{max}})$ if $\mu = 2$. The second moment $M$ is computed likewise:

$$M = \int_0^{\ell_{\text{max}}} \ell^2 \mu(\ell) \, d\ell = \int_0^1 a\ell^2 \, d\ell + \int_1^{\ell_{\text{max}}} a\ell^{2-\mu} \, d\ell.$$  

We have $\int_0^1 a\ell^2 \, d\ell = \frac{a}{3}$ for the first term, and for the second term

$$\int_1^{\ell_{\text{max}}} \ell^{2-\mu} \, d\ell = \begin{cases} 
\Theta(\ell_{\text{max}}^{3-\mu}) & \text{if } \mu < 3 \\
\Theta(\log(\ell_{\text{max}})) & \text{if } \mu = 3
\end{cases}.$$  

Now remark that $\tau^2 = o(M)$, so that $\sigma^2 = \Theta(M)$.

\[\square\]

A.2 On the connection between time and number of steps

To ease the notation, we drop the dependency on $n$ in several notations when it is clear from the context. Recall that we assume that the scan phase in-between ballistic step takes $\tau_{\text{scan}} = O(1)$ time. We next observe, that we may assume without loss of generality that this phase takes zero time, rather than a constant. Indeed, Claim 3 connects the detection time with the expected number of moves times the expected length of a step. If we take into consideration that the duration of the scan phase is $\tau_{\text{scan}} = O(1)$, then we would need to multiply the expected number of moves times the expected length of a step. If we take into consideration that the duration of the scan phase is $\tau_{\text{scan}} = O(1)$, then we would need to multiply the expected number of moves times the average time to take a step (including the pause before it) which is $\tau + \tau_{\text{scan}}$ instead of by $\tau$. As shown in Claim 2, we have $\tau = \Omega(1)$ and thus $\tau + \tau_{\text{scan}} = \Theta(\tau)$. This implies that the asymptotic detection time is not affected by assuming that $\tau_{\text{scan}} = 0$.

Let us denote by $T(m)$ the random time taken by the walk up to step $m$, i.e.

$$T(m) = \sum_{s=1}^{m} \|V(s)\|,$$

where $V(s) = (V_1(s), V_2(s))$ is the vector chosen at step $s$, and $\|V(s)\| = |V_1(s)| + |V_2(s)|$. Let us denote by $m_{\text{detect}}^X(S)$ the random number of steps before $X$ detects $S$ for the first time (i.e., since the searcher has a perception radius 1, $m_{\text{detect}}^X(S)$ is the first $m$ such that $X(m) \in B(S)$). By definition, the expected time before detecting $S$ is $t_{\text{detect}}^X(S) = \mathbb{E}(T(m_{\text{detect}}^X(S)))$. We next argue that this time equals the average number of steps needed to hit $S$, multiplied by the average time $\tau$ needed for one step.

Claim 3. For any intermittent random walk $X$ on $\mathbb{T}_n$, and any set $S \subseteq \mathbb{T}_n$,

$$t_{\text{detect}}^X(S) = \mathbb{E}(m_{\text{detect}}^X(S)) \cdot \tau,$$

where $\tau = \mathbb{E}(\|V(1)\|)$ is the expected step-length.
Claim 3 reminds of Wald’s identity with respect to the lengths $(\|V(s)\|)_s$. However, Wald’s identity cannot be applied directly because $m_{\text{detect}}^X(S)$ is not a stopping step for the sequence $(\|V(s)\|)_s$. (The usual terminology is stopping time, but we employ the term "step" here so as to emphasis that the variable counts steps.) Instead, we prove the claim by the Martingale Stopping Theorem (that can also be used to prove Wald’s identity).

Proof. To prove the claim, note that we can suppose that $\tau < \infty$ and $E(m_{\text{detect}}^X(S)) < \infty$. Indeed, if $\tau = \infty$, then even one step takes an infinite expected time. Moreover, since $p(0) < 1$ by definition, there exist $\varepsilon, \delta > 0$ such that the probability that a length of a step is at least $\varepsilon$ is at least $\delta$. If $E(m_{\text{detect}}^X(S)) = \infty$, then, after $m$ steps, where $m$ is large, there are roughly $\delta m$ steps of length at least $\varepsilon$. Hence, if there is an infinite number of steps, then with probability 1 there is an infinite number of steps, each of which taking time at least $\varepsilon$. In both cases, we have $t_{\text{detect}}^X(S) = \infty$, and the equality is verified. In what follows we therefore assume that both $\tau < \infty$ and $E(m_{\text{detect}}^X(S)) < \infty$. We start the proof by defining:

$$W(m) := \sum_{s \leq m} (\|V(s)\| - \tau).$$

The claim is proven by showing first that $(W(m))_m$ is a martingale with respect to $(X(m))_m$. Then, as $m_{\text{detect}}^X(S)$ is a stopping step for $(X(m))_m$ (i.e., the event $\{m_{\text{detect}}^X(S) = m\}$ depends only on $X(s)$, for $s \leq m$), we can apply the Martingale Stopping Theorem which gives $\sum_{s \leq m} X_{\text{detect}}^X(S)(\|V_s\| - \tau) = 0$. In more details, recall, e.g., from [41] (Definition 12.1), that a sequence of random variables $(W(m))_m$ is a martingale with respect to the sequence $(X(m))_m$ if, for all $m \geq 0$, the following conditions hold:

- $W(m)$ is a function of $X(0), X(1), \ldots, X(m)$,
- $E(|W(m)|) < \infty$,
- $E(W(m+1) \mid X(0), \ldots, X(m)) = W(m)$.

We first claim that $W(m)$ is a martingale with respect to $X(0), X(1), \ldots$. Indeed, since $V(s) = X(s) - X(s-1)$, the first condition holds. Since $E(|W(m)|) \leq \sum_{s \leq m} E(|V_s - \tau|) \leq 2\tau m < \infty$, the second condition holds. Finally, since $W(m+1) = W(m) + \|V(m+1)\| - \tau$, we have $E(W(m+1) \mid X(0), \ldots, X(m)) = W(m) + E(\|V(m+1)\|) - \tau = W(m)$, and hence the third condition holds as well.

Next, recall the Martingale Stopping Theorem (see [41], Theorem 12.2) which implies that $E(W(M)) = E(W(0))$, whenever the following three conditions hold:

- $W(0), W(1), \ldots$ is a martingale with respect to $X(0), X(1), \ldots$,
- $M$ is a stopping step for $X(0), X(1), \ldots$ such that $E(M) < \infty$, and
- there is a constant $c$ such that $E(|W(m+1) - W(m)| \mid X(0), \ldots, X(m)) < c$. 


Let us prove that the conditions of the Martingale Stopping theorem hold. We have already seen that the first condition holds. Secondly, we have \( \mathbb{E}(m_{detect}^X(S)) < \infty \) by hypothesis. Finally, we need to prove that \( \mathbb{E}(|W(m + 1) - W(m)| \mid X(0), \ldots, X(m)) < c \) for some \( c \) independent of \( m \). Since \( W(m + 1) - W(m) = \|V(m + 1)\| - \tau \), we have \( \mathbb{E}(|W(m + 1) - W(m)| \mid X(0), \ldots, X(m)) = \mathbb{E}(\|V(m + 1)\| - \tau) \leq 2\tau \). Therefore, the conditions hold and the theorem gives:

\[
\mathbb{E}(W(m_{detect}(S))) = \mathbb{E}(W(0)) = 0.
\]

Hence,

\[
0 = \mathbb{E}(W(m_{detect}^X(S))) = \mathbb{E}
\left( -m_{detect}^X(S)\tau + \sum_{s \leq m_{detect}^X(S)} \|V(s)\| \right)
\]

\[
= -\mathbb{E}(m_{detect}^X(S))\tau + \mathbb{E}
\left( \sum_{s \leq m_{detect}^X(S)} \|V_s\| \right)
\]

\[
= -\mathbb{E}(m_{detect}^X(S))\tau + t_{detect}^X(S),
\]

which establishes Claim 3.

\[\square\]

### A.3 Monotonicity

A function \( f \) on \( \mathbb{R}^2 \) is called radial if there is a function \( \tilde{f} \) on \( \mathbb{R}^+ \) such that for any \( x \in \mathbb{R}^2 \), \( f(x) = \tilde{f}(\|x\|) \). In this case we say that \( f \) is non-increasing if \( \tilde{f} \) is. The goal of this section is to prove the following.

**Claim 4.** Let \( X \) and \( Y \) be two independent random variables with values in \( \mathbb{R}^2 \), admitting probability density functions respectively \( f \) and \( g \). Let \( h \) be the probability density functions of \( X + Y \). If \( f \) and \( g \) are both radial and non-increasing functions then so is \( h \).

We shall soon prove the claim, but first, let us give a corollary, assuming the claim is true.

**Corollary 5** (Monotonicity). Let \( Z \) be a random walk process on \( \mathbb{R}^2 \), starting at \( Z(0) = 0 \), with step-length distribution \( p \). If \( p \) is non-increasing, then for any \( m \geq 1 \) the distribution \( p^{Z(m)} \) of \( Z(m) \) is radial and non-increasing. In particular, for any \( x, x' \) points in \( \mathbb{R}^2 \) with \( \|x\| \leq \|x'\| \), we have \( p^{Z(m)}(x') \leq p^{Z(m)}(x) \). Furthermore, for any \( x \in \mathbb{R}^2 \) and any \( m \geq 1 \), \( p^{Z(m)}(x) \leq \frac{1}{\pi \|x\|^2} \).

**Proof.** The fact that \( p^{Z(m)} \) is radial and non-increasing follows from Claim 4 by induction. Indeed, the step-length vectors \( V(1), V(2), \ldots \) are independent and, by hypothesis, admit a radial, non-increasing p.d.f. Hence so does \( Z(m) = V(1) + V(2) + \cdots + V(m) \). The upper bound on \( p^{Z(m)}(x) \) follows easily. Indeed, for \( x \in \mathbb{R}^2 \setminus \{(0,0)\} \), consider the ball \( B \) of radius \( \|x\| \) and centered at 0. We have \( \int_B p_m^Z(y)dy \leq 1 \), and by the monotonicity, \( \int_B p_m^Z(y)dy \geq p_m^Z(x)|B| = p_m^Z(x) \cdot \pi \|x\|^2 \). \[\square\]
Proof of Claim 4. Let $\theta \in [0, 2\pi)$. For $x \in \mathbb{R}^2$, denote by $\text{rot}_\theta(x)$ the point obtained by rotating $x$ around the center 0 with an angle of $\theta$. Then, by a change of variable, we have:

$$h(\text{rot}_\theta(x)) = \int_{y \in \mathbb{R}^2} f(\text{rot}_\theta(x) - y)g(y)dy$$

$$= \int_{y \in \mathbb{R}^2} f(\text{rot}_\theta(x) - \text{rot}_\theta(y))g(\text{rot}_\theta(y))dy$$

$$= \int_{y \in \mathbb{R}^2} f(x - y)g(y)dy = h(x),$$

where we used in the last equality the radiality of $f$ and $g$. This establishes the fact that $h$ is radial. Next, we prove, in a manner inspired by Adler et al. [42], that $h(\gamma) = h(\nu\gamma)$ instead of $h(\nu,\gamma)$ for simplicity of notation. Now write, beginning with the change of variable $y_2 \mapsto -y_2$,

$$h(x) = \int_{y_1 \in \mathbb{R}} \int_{y_2 \in \mathbb{R}} H_x(-y_2)dy_1dy_2 = \int_{y_1 \in \mathbb{R}} \int_{y_2 \in \mathbb{R}} H_x(-y_2 - \gamma)dy_1dy_2$$

$$= \int_{y_1 \in \mathbb{R}} \left( \int_{y_2 \geq 0} H_x(-y_2 - \gamma)dy_2 + \int_{y_2 \leq 0} H_x(-y_2 - \gamma)dy_2 \right) dy_1$$

$$= \int_{y_1 \in \mathbb{R}} \int_{y_2 \geq 0} H_x(-y_2 - \gamma)dy_2 + H_x(y_2 - \gamma)dy_2dy_1,$$

and

$$h(x') = \int_{y_1 \in \mathbb{R}} \int_{y_2 \in \mathbb{R}} H_{x'}(y_2)dy_1dy_2$$

$$= \int_{y_1 \in \mathbb{R}} \int_{y_2 \geq 0} H_{x'}(y_2)dy_2 + \int_{y_2 \leq 0} H_{x'}(y_2)dy_2dy_1$$

$$= \int_{y_1 \in \mathbb{R}} \left( \int_{y_2 \geq 0} H_{x'}(y_2 + \gamma)dy_2 + \int_{y_2 \leq 0} H_{x'}(y_2 + \gamma)dy_2 \right) dy_1$$

$$= \int_{y_1 \in \mathbb{R}} \left( \int_{y_2 \geq 0} H_{x'}(y_2 + \gamma) + H_{x'}(-y_2 - \gamma)dy_2 \right) dy_1$$

Hence, we have that $h(x) - h(x')$ is equal to

$$\int_{y_1 \in \mathbb{R}} \int_{y_2 \geq 0} f(-y_1, x_2 + y_2 + \gamma)g(y_1, -y_2 - \gamma) + f(-y_1, x_2 - y_2 + \gamma)g(y_1, y_2 - \gamma)$$

$$- f(-y_1, x_2 - y_2 - \gamma)g(y_1, y_2 + \gamma) - f(-y_1, x_2 + y_2 - \gamma)g(y_1, \gamma - y_2)dy_1dy_2$$

Since $g$ is radial, we have $g(y_1, -y_2 - \gamma) = g(y_1, y_2 + \gamma)$ and $g(y_1, \gamma - y_2) = g(y_1, y_2 - \gamma)$. Furthermore, using that $x_2 + \gamma = x_2' - x_2 - \gamma$, we obtain that $h(x) - h(x')$ is equal to:

$$\int_{y_1 \in \mathbb{R}} \int_{y_2 \geq 0} (f(-y_1, x_2 + y_2 + \gamma) - f(-y_1, x_2 - y_2 + \gamma)) (g(y_1, y_2 + \gamma) - g(y_1, y_2 - \gamma)) dy_1dy_2$$
In this summation, since $x_2 \geq 0$, $\gamma \geq 0$ and $y_2 \geq 0$, we have $|x_2 + y_2 + \gamma| \geq |x_2 - y_2 + \gamma|$ and $|y_2 + \gamma| \geq |y_2 - \gamma|$. Since $f$ and $g$ are non-increasing functions of the distance to 0, both factors of the integrand are non-negative, hence the integrand is non-negative and $h(x) - h(x') \geq 0$. \hfill \Box

A.4 Projections of 2-dimensional Lévy walks are also Lévy

Consider a Lévy walk $Z^\mu$ with parameter $\mu$ on $\mathbb{R}^2$, that has maximal step length $\ell_{\text{max}}$ (including the case $\ell_{\text{max}} = \infty$). It is well-known that the projection of a Lévy walk with parameter $\mu$ on each of the axes is also a Lévy walk with parameter $\mu$. For example, the conservation of the power-law distribution under projection was established by Sims et al. [13]. Nevertheless, in this section, we provide another proof for this fact, for completeness purposes, and also because [13] did not examine the case $\ell_{\text{max}} < \infty$.

Without loss of generality, we may consider only the projection $Z^\mu_1$ on the $x$-axis. Hence, we aim to prove the following.

**Theorem 6.** The projection $Z^\mu_1$ of $Z^\mu$ is a Lévy walk on $\mathbb{R}$ with parameter $\mu$, in the sense that the p.d.f. of the step-lengths of $X^\mu_1$ is $p(\ell) \sim 1/\ell^\mu$, for $\ell \in [1, \ell_{\text{max}}/2]$. Furthermore, the variance of $X^\mu_1$ is

$$\sigma'^2 = \begin{cases} \Theta(\ell_{\text{max}}^3-\mu) & \text{if } \mu \in (1, 3) \\ \Theta(\log \ell_{\text{max}}) & \text{if } \mu = 3 \end{cases}.$$ 

**Proof.** It is clear that $Z^\mu_1$ is also a random walk that moves incrementally, with the increments between $Z^\mu_1(m)$ and $Z^\mu_1(m + 1)$ being the projection $Z_1(m + 1)$ of the chosen 2-dimensional vector $V(m + 1) = Z^\mu(m + 1) - Z^\mu(m)$. These projections are i.i.d. variables as the vectors $(V(m))_m$ are i.i.d. variables, and their signs are $\pm$ with equal probability. Hence, all that needs to be verified is that $l_1 := |V_1(1)|$ has a Lévy distribution with parameter $\mu$.

Let $V$ be one step-length drawn according to a Lévy distribution $p^\mu$. Recall that

$$p^\mu(\ell) = \begin{cases} a_\mu & \text{if } \ell \leq 1 \\ a_\mu \ell^{-\mu} & \text{if } \ell \in [1, \ell_{\text{max}}] \\ 0 & \text{if } \ell \geq \ell_{\text{max}} \end{cases},$$

where $a_\mu$ is the normalization factor, with $a_\mu = \frac{1}{1 + \int_1^{\ell_{\text{max}}} \ell^{-\mu} d\ell} = \frac{1}{1 + \frac{1-\frac{1}{\mu}}{\mu}} \in [1 - \frac{1}{\mu}, 1]$. Hence the distribution of $V = (V_1, V_2) \in \mathbb{R}^2$ is

$$p^V(x) = \frac{1}{2\pi \|x\|} p^\mu(\|x\|) = \begin{cases} \frac{a_\mu}{2\pi} \|x\|^{-1} & \text{if } \|x\| \leq 1 \\ \frac{a_\mu}{2\pi} \|x\|^{-\mu-1} & \text{if } \|x\| \in [1, \ell_{\text{max}}] \\ 0 & \text{if } \|x\| \geq \ell_{\text{max}} \end{cases}.$$  \hfill (4)
For $x_1 \in (0, \ell_{\text{max}})$, we have

$$p_{l^1}(x_1) = 2 \int_0^{x_1} \frac{\sqrt{\ell_{\text{max}}^2 - x_1^2}}{\pi} p_V(x_1, x_2) dx_2 \tag{5}$$

$$= \frac{2a_\mu}{\pi} \int_0^{\sqrt{\ell_{\text{max}}^2 - x_1^2}} \frac{1}{1 \|x\| < 1 \|x\| \leq 1 \|x\| \geq 1} dx_2,$$

where $x = (x_1, x_2)$. If $|x_1| \geq 1$, then $\|x\| \geq 1$ for any $x_2 \in \mathbb{R}$, so that

$$p_{l^1}(x_1) = a_\mu \frac{\pi}{x_1^\mu} I(x_1),$$

where

$$I(x_1) := \int_0^{\sqrt{\ell_{\text{max}}^2 - x_1^2}} \frac{1}{(1 + y^2)^{\frac{1+\mu}{2}}} dy.$$

For any $x_1 \in (1, \ell_{\text{max}})$, we have $I(x_1) \leq \int_0^1 \frac{1}{(1 + y^2)^{\frac{1+\mu}{2}}} dy = O(1)$ since $\frac{1}{(1 + y^2)^{\frac{1+\mu}{2}}} = \Theta(y^{-\mu})$, for large $y$, and this function of $y$ is integrable as $\mu > 1$. Furthermore, if $|x_1| \leq \ell_{\text{max}}/2$, we have $I(x_1) \geq \int_0^1 \frac{1}{(1 + y^2)^{\frac{1+\mu}{2}}} dy$ which is a positive constant. Hence, if $|x_1| \in (1, \ell_{\text{max}}/2)$, we have

$$p_{l^1}(x_1) = \Theta \left( \frac{1}{x_1^\mu} \right), \tag{5}$$

and for $\ell_{\text{max}}/2 \leq x_1 \leq \ell_{\text{max}}$, we have

$$p_{l^1}(x_1) = O \left( \frac{1}{x_1^\mu} \right). \tag{6}$$

Hence, the projection of the Lévy walk on the axes are Lévy-like, in the sense that their step-lengths distributions generally follow a power-law of same exponent $\mu$. The expected length, second moment and variance of one projected step are computed as in Claim $\text{\ref{A}}$. Indeed write, for $i \in \{1, 2\}$,

$$\int_0^{\ell_{\text{max}}} x_1^i p_{l^1}(x_1) dx_1 = \Theta \left( \int_0^1 x_1^i p_{l^1}(x_1) dx_1 + \int_1^{\ell_{\text{max}}/2} x_1^{i-\mu} dx_1 + \int_{\ell_{\text{max}}/2}^{\ell_{\text{max}}} x_1^i p_{l^1}(x_1) dx_1 \right).$$

We have $\int_0^{\ell_{\text{max}}} x_1^i p_{l^1}(x_1) dx_1 \leq 1$. Also, it is easy to verify from Eq. (5) and (6) that the third term is dominated by the second term, which in turn, is $\Theta(\int_1^{\ell_{\text{max}}/2} x_1^{i-\mu} dx_1)$. Hence, the expected length, second moment and variance of one projected step are of the same order as those of the non-projected steps given by Claim $\text{\ref{A}}$ which concludes the proof of Theorem 6.
B Lower Bounds

B.1 Random walk with a fixed step-length

In order to illustrate the definition of the overrun, we provide here a simple computation of the overrun of the intermittent process $X$ in which all step lengths are some pre-determined fixed integer $\ell$. Note that the case $\ell = 1$ corresponds to the simple random walk, and that taking $\ell = \Theta(\sqrt{n})$ may be viewed as a ballistic strategy. Consider a disc target of diameter $D < \sqrt{n}/2$. Since the searcher starts at a random point, with constant probability, the target is located at a distance of at least $\sqrt{n}/4$ from the initial location of the searcher. In this case, merely traversing this distance by the random walk process requires $\Omega((\sqrt{n/\ell})^2) = \Omega(n/\ell^2)$ steps on expectation, and hence consumes $\Omega(n/\ell)$ time on expectation. This implies that $\text{Over}^X(n,D) = \Omega(D/\ell)$. Furthermore, as illustrated in the main text (Fig. 1b), and as shown formally in the next section, there are $\Omega(n/D^2)$ possible locations of the target. Since the agent must, on average, visit at least half of those, it will overall need $\Omega(n\ell/D^2)$ time to find the target on expectation, since each step takes $\ell$ time. Thus, we also have $\text{Over}^X(n,D) = \Omega(n/D)$. Altogether, these arguments imply that $\text{Over}^X(n,D) = \Omega(\max\{\ell/D, D/\ell\})$. While $\ell$ can be tuned to optimize the overrun with respect to a specific value of $D$, if we know only an upper bound $D_{\text{max}}$ on the value of $D$ then the overrun would be large with respect to either $D = 1$ or $D = D_{\text{max}}$. Specifically, for $D = 1$ we have $\text{Over}^X(n,1) = \Omega(\ell)$, while for $D = D_{\text{max}}$, we have $\text{Over}^X(n,D_{\text{max}}) = \Omega(D_{\text{max}}/\ell)$. Hence, for at least one value of $D$ among the two, we have $\text{Over}^X(n,D) = \Omega(\sqrt{D_{\text{max}}})$. In particular, if $D_{\text{max}} = n^\delta$ for some $\delta > 0$ then the overrun is polynomial in $n$.

B.2 General lower bounds

We prove here a general proposition that holds for any search process $X$ on the torus whose speed is constant (i.e., it takes $O(\ell)$ units of time to do a ballistic step of length $\ell$). We may assume without loss of generality that the speed is normalized to 1. Note also that, since we aim at a lower bound, we can suppose, without loss of generality, that the scan time in-between steps is 0.

We next define a quantity, termed $T_d$, which will be used to lower bound the time needed to detect an extended target $B(S)$ at distance $d$ or more. Formally, we distinguish between two cases, according to the given process $X$.

- If $X$ is an intermittent random walk, we let $T_d$ be the expected time needed before the end point of a step is at distance at least $d$ from the initial location.

- Otherwise, we simply define $T_d = d$.

Claim 7. Let $X$ be any search process on the torus. Consider any target $S$ of diameter $D < \sqrt{n}/6 - 1$. The expected time to detect $S$ is $\Omega(nT_{D/2})$.

Proof. Consider a target $S$ of diameter $D$ and of an arbitrary shape. Instead of considering that $S$ is fixed and that the initial location $X(0)$ is chosen u.a.r, we may assume without loss of generality
that \( X(0) \) is fixed, say at the origin, and that the center of mass \( u^* \) of \( S \) is chosen uniformly at random in the torus.

Let us first construct a grid with \( s \times s \) nodes, where \( s = \lfloor \sqrt{n}/(3D + 2) \rfloor \). Note that since \( D < \sqrt{n}/6 - 1 \), we have \( s \geq 2 \). To make the grid symmetric, we let the distance between two neighboring nodes be precisely \( \sqrt{n}/s \). We next align the grid so that \( u^* \) is a node of the grid, and construct a disc of radius \( D + 1 \) around each node. Note that the number of discs is \( M = s^2 = \Omega(n/D^2) \), and that the distance between any two discs is at least \( D \). See Figure 1(b) in the main text. Furthermore, note that the disc \( U^\star \) corresponding to \( u^* \) fully contains the extended target \( B(S) \). Let us therefore lower bound the time until visiting \( U^\star \) for the first time. This will serve as the desired lower bound for detecting \( S \).

Assume that the information about the collection of discs is given to the searcher. We may assume this, since it can only decrease the best detection time. Because the location of \( S \) in chosen uniformly at random in the torus, from the perspective of the searcher, each of the discs has an equal probability to be \( U^\star \). It follows that with probability \( 1/2 \), at least half of the discs are visited, before the searcher visits \( U^\star \). Since the discs are separated by distance of at least \( D \), we immediately get that the expected time until visiting \( U^\star \) is \( \Omega(MD) = \Omega(n/D) \), which is the desired claim when \( X \) is not an intermittent random walk (and hence \( T_D = D \)).

Let us next consider the case that \( X \) is an intermittent random walk. The arguments are similar, yet slightly more subtle. We aim to lower bound the time until visiting \( U^\star \) for the first time, where by visiting a disc, we mean that the end of a ballistic step of \( X \) is in that disc. For this purpose, we may assume that the process terminates when it visits \( U^\star \). Let \( U_1, U_2, \ldots \) denote the newly visited discs, in order of visitation, with all the \( U_i \) distinct. Let \( A_i \) be the event that \( U^\star \notin \{U_1, \ldots, U_i\} \). Note that \( \Pr(A_i) = 1 - \frac{i}{M} \). Let \( t_i \) denote the time from visiting \( U_i \) (for the first time) until visiting \( U_{i+1} \) (for the first time), in the event that \( A_i \) occurs. If the event \( A_i \) does not occur, we say that \( t_i = 0 \). The time before visiting \( U^\star \) can therefore be written as \( \sum_{i=1}^{M-1} t_i \). Furthermore, we have \( \mathbb{E}(t_i) = \mathbb{E}(t_i \mid A_i) \Pr(A_i) \). Hence, the expected time before visiting \( U^\star \) is:

\[
\sum_{i=1}^{M-1} \mathbb{E}(t_i \mid A_i) \Pr(A_i).
\]

Now recall that \( X \) is an intermittent Markovian process, and that \( A_i \) corresponds to an event that is relevant up to (and including) the detection of \( U_i \). Hence, \( \mathbb{E}(t_i \mid A_i) \) is lower bounded by the minimal expected time that the intermittent random walk \( X \), starting at some point \( u \in U_i \), visits another disc, where the minimization is taken w.r.t \( u \in U_i \). Since discs are separated by distance of at least \( D \), the process starting at any such \( u \) needs to visit a disc at distance at least \( D \). It therefore follows that \( \mathbb{E}(t_i \mid A_i) \geq T_D \). Altogether, the expected time to detect \( S \) is at least:

\[
\sum_{i=1}^{M-1} T_D \Pr(A_i) = \sum_{i=1}^{M-1} T_D (1 - i/M) = \Omega(T_D M) = \Omega \left( n \frac{T_D}{D^2} \right),
\]

as desired.
**Corollary 8.** For every $1 \leq D \leq \sqrt{n}/2$, the best possible detection time is $\Theta(n/D)$, when we allow the strategy to have continuous detection, to be unrestricted in terms of its internal computational power and navigation abilities, and to be fully tuned to the diameter. In other words, $\text{opt}(n, D) = \Theta(n/D)$.

**Proof.** The fact that $\text{opt}(n, D) = \Omega(n/D)$ for every $D < \sqrt{n}/6 - 1$ follows immediately from Claim 7 and the fact that $T_D \geq D$. For $\sqrt{n}/6 - 1 < D \leq \sqrt{n}/2$ the bound $\Omega(n/D) = \Omega(\sqrt{n})$ follows simply because with constant probability, the target is at distance $\Omega(\sqrt{n})$ from the initial location of the searcher.

In order to see why $\text{opt}(n, D) = O(n/D)$, let us tile the torus with horizontal and vertical lines partitioning the torus into squares of size $D/2 \times D/2$ each. In the case that $\sqrt{n}$ is not a multiple of $D/2$, we might have few of these squares smaller than $D/2 \times D/2$. It is clear that this can be constructed while maintaining that the number of horizontal and vertical lines is $O(\sqrt{n}/D)$. For any connected target $S$ of diameter $D$, the set $B(S)$ must intersect at least one of these lines. Now consider a deterministic strategy that repeatedly walks over this tiling exhaustively, without doing much repetition in each exhaustive search. E.g., by first walking on the horizontal lines exhaustively (with occasional steps to move between horizontal lines) and then walking on the vertical lines exhaustively. It is easy to see that such a strategy exists and requires at most $O(\sqrt{n}/D \cdot \sqrt{n}) = O(n/D)$ time to pass over all the lines, and hence to detect the target. This establishes the required upper bound.

Claim 7, applied with $D = 1$, also yields the following corollary, by remarking that for intermittent random walk processes, $T_D$, namely, the expected time until the end point of a step is at a distance of at least $D$ is at least the expected time for one step $\tau$, i.e., $T_D \geq \tau$.

**Corollary 9.** Consider an intermittent random walk strategy $X$ on the torus $T_n$. The detection time of any target of diameter $D$ is $\Omega(n\tau/D^2)$.

**Claim 10.** Consider a random walk process $X$ on the torus $T_n$ and let $\sigma'$ denote the standard deviation of the length of the projected steps onto either coordinate.

- The expected maximal distance of $X$ to its origin after $m$ steps, i.e. $\max_{s \leq m} \|X(s) - X(0)\|$, is $O(\sqrt{m}\sigma')$.

- Let $m_d$ be the number of steps needed to go to distance at least $d < \sqrt{n}/2$, in other words $m_d$ is the first step $m$ for which $\|X(m) - X(0)\| \geq d$. We have $E(m_d) = \Omega(d^2/\sigma'^2)$.

- If the process is intermittent and $\tau$ denotes the average length of a jump, then the expected time before reaching distance $d < \sqrt{n}/2$ is $T_d = \Omega(d^2/\sigma'^2\tau)$.

In particular, if the process is intermittent and $L$ is the maximal length in the support of the step-length distribution, then the expected time needed to go to a distance $\Omega(\sqrt{n})$ is $\Omega(nL)$.

We will use Claim 10 in the next section to get an upper bound on the time needed for a Lévy walk to reach some distance. The proof of Claim 10 is based on Kolmogorov’s inequality.
Proof. Let $Z$ be the process on $\mathbb{R}^2$, with $Z(0) = X(0)$ and evolving with the same steps as $X$. Since the distance between $Z(m)$ and $Z(0)$, in $\mathbb{R}^2$, is always at least that of $X(m)$ and $X(0)$, in $T_n$, the number of steps needed to go to distance $d$ in $T_n$ is at least as high as in $\mathbb{R}^2$. Hence, we may analyze the process $Z$ instead of $X$.

Define $d^Z_{\max}(m)$ as the maximal distance (from the initial point) that the process $Z$ reached from step 0 up to step $m$, i.e.,

$$d^Z_{\max}(m) = \max_{s \leq m} \|Z(0) - Z(s)\|.$$ 

Now write $Z = (Z_1, Z_2)$, let $p'$ be the p.d.f. of the projected step-lengths (i.e. the p.d.f. of the step-lengths of $Z_i$), and let $\tau'$ and $\sigma'$ be respectively its mean and standard deviation. Next, let $d^{Z}_{i,\max}(m)$ be the maximal distance reached by the projection on coordinate $i = 1, 2$. Since steps are independent, the standard deviation of $Z_i(s)$, for $s \leq m$, is $\sqrt{s}\sigma' \leq \sqrt{m}\sigma'$.

By Kolmogorov’s inequality, we have for any $\lambda > 0$, $\Pr(d^Z_{i,\max}(m) \geq \lambda\sqrt{m}\sigma') \leq \frac{1}{\lambda^2}$. Furthermore, since $d^Z_{\max}(m) \leq \sqrt{2}\max\{d^Z_{1,\max}(m), d^Z_{2,\max}(m)\}$, we have by a union bound argument, for any $\lambda > 0$,

$$\Pr(d^Z_{\max}(m) \geq \lambda\sqrt{m}\sigma') \leq \Pr\left(d^Z_{1,\max}(m) \geq \frac{\lambda}{\sqrt{2}}\sqrt{m}\sigma'\right) + \Pr\left(d^Z_{2,\max}(m) \geq \frac{\lambda}{\sqrt{2}}\sqrt{m}\sigma'\right) \leq \frac{4}{\lambda^2}. \quad (7)$$

Hence,

$$\mathbb{E}(d^Z_{\max}(m)) = \int_{s=0}^{\infty} \Pr\left(d^Z_{\max}(m) \geq s\right) ds \leq \sum_{\lambda' = 0}^{\infty} \int_{\lambda' = 0}^{\sqrt{m}\sigma'} \Pr\left(d^Z_{\max}(m) \geq \lambda\sqrt{m}\sigma' + \lambda'\right) d\lambda' \leq \sqrt{m}\sigma' \sum_{\lambda' = \lambda' \geq 0} \Pr(d^Z_{\max}(m) \geq \lambda\sqrt{m}\sigma') = O\left(\sqrt{m}\sigma'\right), \quad (8)$$

which proves the first item of Claim [10]

Next, write the $m_d$ of the statement as $m^X_d$, to distinguish it from the similarly defined $m^Z_d$, which is the first step for which $\|Z(m) - Z(0)\| \geq d$. As remarked above, we have $m^X_d \geq m^Z_d$. Note that for $m \geq m^Z_d$, we have $d^Z_{\max}(m) \geq d^Z_{\max}(m_d) \geq d$. Therefore, by Markov’s inequality,

$$\mathbb{E}(d^Z_{\max}(2\mathbb{E}(m^Z_d))) \geq \mathbb{E}(d^Z_{\max}(2\mathbb{E}(m^Z_d)) \mid m^Z_d < 2\mathbb{E}(m^Z_d)) \cdot \Pr(m^Z_d < 2\mathbb{E}(m^Z_d)) \geq d \cdot \frac{1}{2}. \quad (9)$$

Now using Eq. (8) with $m = 2\mathbb{E}(m^Z_d)$, we have $\mathbb{E}(d^Z_{\max}(2\mathbb{E}(m^Z_d))) = O(\sqrt{\mathbb{E}(m^Z_d)}\sigma')$ and hence, by Eq. (9),

$$\mathbb{E}(m^X_d) \geq \mathbb{E}(m^Z_d) = \Omega\left(\frac{d^2}{\sigma'^2}\right),$$

which proves the second item of Claim [10]

The last item is a lower bound on $T_d = \mathbb{E}(T(m^X_d))$, the expected time that $X$ needs to reach distance $d$. To obtain it, we observe that $m^X_d$ is the hitting step of the set of nodes at distance $d$ or more in the torus. Hence, by Claim [3] we have $T_d = \mathbb{E}(m^X_d) \cdot \tau \geq \mathbb{E}(m^Z_d) \cdot \tau = \Omega(\frac{d^2}{\sigma'^2}\tau)$, which was exactly as needed.
Finally, observe that
\[ \sigma^2 = \int_0^L p'(\ell)\ell^2 d\ell \leq \int_0^L p'(\ell)\ell \cdot L d\ell = L\tau' \leq L\tau, \tag{10} \]
where the last inequality is justified by the fact that the projection reduces distances. This completes the proof of Claim \[10\].

### B.3 Lower bounds for Lévy walks

The goal of this section is to prove lower bounds on the overrun of Lévy walks other than Cauchy. For \(1 < \mu < 2\), we show that the corresponding intermittent Lévy walks are bad at finding small targets. For \(2 < \mu \leq 3\), we show that the corresponding Lévy walks are bad at finding large targets.

#### B.3.1 Intermittent Lévy walks with \(1 < \mu \leq 2\)

Let \(X^\mu\) be the intermittent Lévy walk on the torus \(T_n\), for some \(1 < \mu < 2\). We start by analyzing the detection times of small targets.

**Theorem 11.** Let \(\mu \in (1, 2)\) and \(D \in [1, \sqrt{n}/2]\). Write \(\mu = 2 - \varepsilon\). The detection time of the Lévy walk \(X^\mu\) with respect to a target \(S\) of diameter \(D\) is
\[ t^X_{\text{detect}}(S) = \Omega(n^{1+\varepsilon/2}/D^2), \tag{11} \]
and the overrun w.r.t. \(D\) is:
\[ \text{Over}^X(n, D) = \Omega(n^{\varepsilon/2}/D). \tag{12} \]

**Proof.** By Corollary 9, the detection time of a target \(S\) with diameter \(D\) is \(\Omega(n\tau/D^2)\) where \(\tau\) is the expected step length. Using that \(\ell_{\text{max}} = \Theta(\sqrt{n})\), Claim 2 implies that this expected step length is, for \(\mu = 2 - \varepsilon\) with \(\varepsilon \in (0, 1)\):
\[ \tau = \Theta(n^{1-\mu/2}) = \Theta(n^{\varepsilon/2}). \]
Hence, the detection time \(X^\mu\) for a target of diameter \(D\) is \(\Omega(n^{1+\varepsilon/2}/D^2)\). Dividing this by the unconditional optimal time \(\Theta(n/D)\), we get the desired lower bound on the overrun. \(\square\)

#### B.3.2 Lévy walks with \(2 < \mu \leq 3\)

Theorem 11 implies that the overrun of the intermittent Lévy walk \(X^\mu\) for \(\mu \in (1, 2)\) is very large with respect to small targets, i.e, when \(D \ll n^{\varepsilon/2}\). We next aim to prove the case \(\mu \in [2, 3]\):

**Theorem 12.** Let \(\mu \in (2, 3]\) and \(D \in [2, \sqrt{n}/6 - 1]\). Write \(\mu = 2 + \varepsilon\) where \(0 < \varepsilon \leq 1\). The following holds with respect to the Lévy process \(X^\mu\) whether it is intermittent or not. The detection time of \(X^\mu\) with respect a target \(S\) of diameter \(D\) is
\[ t^X_{\text{detect}}(S) = \begin{cases} \Omega(nD^{\varepsilon-1}) & \text{if } \mu = 2 + \varepsilon, \text{ where } 0 < \varepsilon < 1, \\ \Omega(\frac{n}{\log D}) & \text{if } \mu = 3. \end{cases} \]
Hence, the overrun of $X^\mu$ with respect to $D$ is:

$$\text{Over}^{X^\mu}(n, D) = \begin{cases} 
\Omega(D^\varepsilon) & \text{if } \mu = 2 + \varepsilon, \text{ where } 0 < \varepsilon < 1, \\
\Omega(D^{\varepsilon/\log D}) & \text{if } \mu = 3.
\end{cases}$$

Since the proof is simpler, let us first prove Theorem 12 for the intermittent setting, i.e., targets can only be detected in-between steps.

**Proof of Theorem 12 for the intermittent setting.** Towards proving the theorem, we first establish the following.

**Claim 13.** Let $X^\mu$ be an intermittent Lévy walk process on the torus $\mathbb{T}_n$, for $\mu \in [2, 3]$, with $\ell_{\text{max}} = \sqrt{n}/2$. The expected time required to reach a distance of $d \geq 1$ from the starting point is:

$$T_d = \begin{cases} 
\Omega(d \log d) & \text{if } \mu = 2, \\
\Omega(d^{\mu-1}) & \text{if } \mu \in (2, 3), \\
\Theta(d^2 \log d) & \text{if } \mu = 3.
\end{cases}$$

**Proof.** We may suppose that $d \in [1, \sqrt{n}/4]$. Denote by $m_d$ the random number of steps before the process reaches a distance of at least $d$. Let us define $m_0 = \lceil d^{\mu-1} \rceil$, and say that a step is small if it has length at most $d$. Define the event $\mathcal{A}$ that all the steps $1, 2, \ldots, m_0$ are small. Note that since $d \leq \ell_{\text{max}}/2$, the probability for any given step not to be small is

$$q = \int_{d}^{\ell_{\text{max}}} \frac{\mu}{\ell^{\mu-1}} d\ell \geq c \frac{d^{\mu-1}}{d^{\mu-1}}$$

for some constant $c \in (0, 1)$. Hence, the probability for a step to be small is $1 - q$, and since the steps are independent, we have:

$$\Pr(\mathcal{A}) = (1 - q)^{m_0} = \exp(m_0 \log(1 - q)) \geq \exp(d^{\mu-1} \log(1 - cd^{1-\mu})).$$

We have:

$$\exp(d^{\mu-1} \log(1 - cd^{1-\mu})) = \exp(d^{\mu-1}(-cd^{1-\mu} + o(d^{1-\mu}))) = \exp(-c + o(1)),$$

which is a positive constant. Since this is a continuous, strictly positive, function of $d \in [1, \infty)$, we have $\Pr(\mathcal{A}) \geq c'$ for some constant $c' > 0$ independent of $d$.

Next, note that

$$\mathbb{E}(T(m_d)) \geq \Pr(\mathcal{A}) \cdot \mathbb{E}(m_d \mid \mathcal{A}) = c' \cdot \mathbb{E}(T(m_d) \mid \mathcal{A}).$$

Hence, for the purposes of obtaining a lower bound, it is sufficient to examine the process when conditioned on $\mathcal{A}$. This is a Lévy process of parameter $\mu$, with cut-off $\ell_{\text{max}} = d$. The expected length $\tau$ of a jump is given by Claim 2

$$\tau = \Theta(1) \quad (13)$$

and the variance $\sigma^2$ of the step-length of a jump projected onto one of the axes is given by Theorem 6

$$\sigma^2 = \begin{cases} 
\Theta(d^{3-\mu}) & \text{if } \mu \in (1, 3), \\
\Theta(\log d) & \text{if } \mu = 3.
\end{cases}$$
To conclude, we use Claim 10:

$$T_d = \Omega \left( \frac{d^2}{\sigma^2} \cdot \tau \right) = \begin{cases} 
\Omega(d^{\mu-1}) & \text{if } \mu \in (2, 3) \\
\Omega(d^2) & \text{if } \mu = 3 
\end{cases}.$$ 

This concludes the proof of Claim 13. \qed

Combining Claim 13 with the fact that the expected time to detect a target of diameter $D$ is $\Omega(nT_D/D^2)$, as established by Claim 7, and comparing to the unconditional optimal detection time $\Theta(n/D)$ for targets of diameter $D$, Theorem 12 is proved in the intermittent case. Next, we prove the theorem when the process is able to detect the target while moving.

**Proof of Theorem 12 for the continuous detection model.** Recall, from the proof of Claim 7 that we can build a grid of $M = \Theta(n/D^2)$ discs of diameter $D$, one of which contains the target, and separated by distance $D$. Furthermore, for every strategy, whether intermittent or not, with probability $\frac{1}{2}$, at least half of the discs are visited before finding the target. Hence, the expected time to find the target is at least half of the expected time to visit half of the discs. In the remaining of the proof we aim to lower bound the expected time to visit half of the discs.

Let $\mu > 2$ and write $\mu = 2 + \varepsilon$. Define a step to be large if it has length $D$ or more. Divide the execution into a sequence of consecutive phases, so that each phase is a succession of small steps, and a final large step (possibly, there are no small steps in the phase if two large steps are consecutive). In short, in what follows we prove that a phase visits $O(1)$ discs on average when $2 < \mu < 3$, or $O(\log D)$ for $\mu = 3$ (Lemma 14), and lasts, on average, $\Omega(D^{\mu-1})$ time (Lemma 17). We then conclude that, after $R = \tilde{\Theta}(M)$ phases, with constant probability, no more than $M/2$ discs are visited and the time spent is

$$\tilde{\Omega}(MD^{\mu-1}) = \tilde{\Omega}(nD^{\mu-3}) = \tilde{\Omega}(nD^{\varepsilon-1}).$$

A straightforward computation then allows to establish the desired bound on the overrun of the Lévy search in the continuous detection model.

We next proceed to explain the proof in details. Let $N_{\text{discs}}$ be the number of discs visited during a phase.

**Lemma 14.** $\mathbb{E}(N_{\text{discs}}) = \begin{cases} 
O(1) & \text{if } 2 < \mu < 3 \\
O(\log D) & \text{if } \mu = 3 
\end{cases}.$

**Proof of Lemma 14.** Given a phase, by linearity of expectation, $\mathbb{E}(N_{\text{discs}})$ equals the expected number of discs visited by the small steps of the phase plus the expected number of discs visited by the large step. The latter quantity is easy to bound. Indeed, since discs are separated by a distance of $D$, the number of discs visited in a step of length $L$ is $O(1 + L/D)$. Moreover, it is easy to verify that, as $\mu > 2$, the expected length of a large step is $\Theta(D)$. Hence the expected number of discs visited during the large step of a phase is $O(1)$. 


In the remaining of the proof of Lemma 14, we aim to upper bound the expected number of discs visited by the small steps of the phase.

Let $D_{\text{small}}$ denote the number of discs discovered during the small steps. Towards establishing an upper bound on $\mathbb{E}(D_{\text{small}})$, let $\alpha$ be the probability for one step to be large. This equals $\alpha \int_{\ell=D}^{\ell_{\text{max}}} \ell^{-\mu} d\ell = \frac{\alpha}{\mu-1}(D^{1-\mu} - \ell_{\text{max}}^{1-\mu})$, and so, as $D < \ell_{\text{max}}/2 = \sqrt{n}/4$, we have:

$$\alpha = \Theta(D^{1-\mu}).$$

Let $N_{\text{small}}$ be the total number of small steps in one phase. Since a phase ends after performing a long step for the first time, we have, for every integer $m \geq 0$, $\Pr(N_{\text{small}} = m) = \alpha(1 - \alpha)^m$. We thus have:

$$\mathbb{E}(D_{\text{small}}) = \sum_{m \geq 0} \alpha(1 - \alpha)^m \cdot \mathbb{E}(D_{\text{small}} \mid N_{\text{small}} = m).$$

**Claim 15.** For any integer $m$, $\mathbb{E}(D_{\text{small}} \mid N_{\text{small}} = m) = O(1 + m\sigma''^2/D^2)$, where $\sigma''$ is the standard deviation of the length of a small step, when projected on one of the coordinates.

Note that the direction of each step is chosen uniformly at random, hence $\sigma''$ does not depend on which coordinate is chosen.

**Proof of Claim 15.** Let $W_i$ be the number of steps before a distance of $2D$ from the initial location is first reached. For $r \geq 1$, define recursively both $S_r = \sum_{i=1}^{r} W_i$, and $W_{r+1}$ to be the number of steps before we first have $\|X(S_r + W_{r+1}) - X(S_r)\| \geq 2D$. Note that the $(W_i)_i$ are i.i.d and have the same law as $m_{2D}$. Hence, by Claim 14 we have

$$\mathbb{E}(W_i) = \Omega(D^2/\sigma''^2).$$

For a given $m \geq 1$, let $r(m)$ be the first $r \geq 1$ for which $S_r > m$ (if this never happens then $r(m) = 0$). Because in-between steps $W_i$ and $W_{i+1}$ only a distance $O(D)$ is travelled, there can only be $O(1)$ discs visited during this time interval. Hence, up to step $m$, at most a number $O(1 + r(m))$ discs are visited. We are thus looking for an upper bound on $\mathbb{E}(r(m))$.

Observe that $r(m)$ is a stopping time for the $(W_i)_{i \geq 1}$. Furthermore, $r(m) \leq m$ since $W_i \geq 1$ for all $i$. Since the $W_i$ are i.i.d., and $\mathbb{E}(W_1)$ is finite also, we can apply Wald’s equation (see [41], Theorem 12.3) to obtain $\mathbb{E}(r(m)) \mathbb{E}(W_1) = \mathbb{E}(S_{r(m)})$, and hence:

$$\mathbb{E}(r(m)) = \frac{\mathbb{E}(S_{r(m)})}{\mathbb{E}(W_1)}.$$  

Moreover, we have $\mathbb{E}(S_{r(m)}) = \mathbb{E}(S_{r(m)-1}) + \mathbb{E}(W_{r(m)})$. By definition of $r(m)$, we have $\mathbb{E}(S_{r(m)-1}) \leq m$. Next, we wish to bound $\mathbb{E}(W_{r(m)})$. Note that $W_{r(m)}$ is at most the first $r > m$ for which $\|X(r) - X(m)\| \geq 4D$. Indeed, by definition of $r(m)$ we have $\|X(m) - X(S_{r(m)-1})\| \leq 2D$ and $\|X(S_{r(m)}) - S_{r(m)-1}\| \geq 2D$. Hence we have $\mathbb{E}(W_{r(m)}) \leq \mathbb{E}(m_{4D}) + m$. Furthermore, we claim that $\mathbb{E}(m_{4D}) = O(\mathbb{E}(m_{2D}))$. Indeed, consider a circle of radius $4D$ from the initial location and a step $s$, for which the agent is within the circle. Consider $E = \mathbb{E}(S_2) = 3\mathbb{E}(m_{2D})$. Starting at step $s$, with
constant probability, there exists three steps \( s_1, s_2, s_3 \in (s, s + 2E) \) for which \( \|X(s_i) - X(s_{i+1})\| \geq 2D \). Furthermore, whenever this happens, a distance of at least \( 4D \) from the center of the circle will be reached if \( X(s_1), X(s_2) \) and \( X(s_3) \) are aligned approximately in the direction leading to the shortest exit from the circle, which happens with constant probability. Hence, after \( 2E \) steps from any step \( s \) where the agent is within the circle, with constant ability, the walk escapes the circle. Applying this argument repeatedly implies that, \( E(m_{4D}) = O(\log D) = O(E(m_{2D})) \). Altogether, we deduce that

\[
E(r(m)) = O(1 + m/E(m_D)) = O(1 + m\sigma'^2/D^2).
\]

As remarked above, up to step \( m \), there are at most \( O(1 + r(m)) \) visited discs. Hence, conditioning on \( N_{small} = m \), there are only \( O(1 + m\sigma'^2/D^2) \) discs visited in the small steps phase, on expectation. This completes the proof of Claim [15].

Using Claim [15] we return to Eq. (14), to bound the expected number of discs visited in a small phase:

\[
E(D_{small}) = \sum_{m \geq 0} \alpha(1 - \alpha)^m \cdot O(1 + m\sigma'^2/D^2) = O(1) + O(\sigma'^2\alpha/D^2 \cdot \alpha^{-2}),
\]

where we used that \( \sum_{m \geq 0} (1 - \alpha)^m = \alpha^{-1} \), and that \( \sum_{m \geq 0} m(1 - \alpha)^m = O(\alpha^{-2}) \). Thus,

\[
E(D_{small}) = O(1 + \sigma'^2\alpha^{-1}/D^2). \quad (17)
\]

As \( \sigma'^2 \) is the variance of the projected Lévy distribution with cut-off \( \ell_{max} = D \), it is given by Theorem [6] as: \( O(D^{3-\mu}) \) for \( \mu < 3 \) and \( O(\log D) \) for \( \mu = 3 \). Together with the fact that \( \alpha = \Theta(D^{1-\mu}) \), we get that the expected number of discs visited by the small steps of a phase is \( O(1) \) for \( \mu \in (2, 3) \) and \( O(\log D) \) for \( \mu = 3 \). Combining with the expected number of discs visited by the large step, which was shown to be \( O(1) \), the proof of Lemma [14] is complete.

Given a constant \( \tilde{c} \), define the following quantity that will refer to the number of phases.

\[
R = \begin{cases} 
\tilde{c}M \text{ if } \mu \in (2, 3) \\
\tilde{c}M/\log D \text{ if } \mu = 3 
\end{cases} . \quad (18)
\]

Given \( R \), let \( N_{discs}^R \) denote the total number of discs visited by the end of the \( R \)-th phase.

**Lemma 16.** For any \( \delta < 1 \), there exists a constant \( \tilde{c} > 0 \) such that the probability to have visited at most \( M/2 \) discs after \( R \) phases (as defined in Eq. (18)) is

\[
\Pr(N_{discs}^R < M/2) > \delta.
\]

**Proof of Lemma 16.** Note that steps are independent and, hence, phases are independent, implying that the number of discs visited during a phase does not depend on the phase number. We have, by linearity of expectation, \( E(N_{discs}^R) = R \cdot E(N_{discs}) \), and, by Markov’s inequality, we have

\[
\Pr(N_{discs}^R \geq M/2) \leq 2R \cdot E(N_{discs})/M.
\]

By Lemma [14] \( E(N_{discs}) \leq c \) for \( \mu \in (2, 3) \) and \( E(N_{discs}) \leq c \log D \) for \( \mu = 3 \), for some constant \( c > 0 \). Hence, we find that \( 2R \cdot E(N_{discs})/M \) is at most \( 2c\tilde{c} \), which can be made to be less than \( 1 - \delta \) by choosing \( \tilde{c} < (1 - \delta)/(2c) \).
Lemma 17. Let $T^R$ be the time spent during $R$ phases. There are two constants $c > 0$ and $q > 0$ for which

$$\Pr(T^R \geq cD^{\mu-1}R) \geq q.$$  

Proof of Lemma 17. Define a phase to be long if it lasts at least $T^\star = c_1D^{\mu-1}$ time for some constant $c_1$ to be fixed later. Let $N_{\text{long-phases}}^R$ be the number of long phases, up to the $R$-th one. Note that

$$T^R \geq T^\star N_{\text{long-phases}}^R.$$  

Let $T$ be the time duration of the small steps in a phase. Since phases are independent, we have:

$$\mathbb{E}(N_{\text{long-phases}}^R) = R \cdot \Pr(T \geq T^\star) \geq R \cdot \Pr(N_{\geq \frac{1}{2}} \geq 2T^\star),$$

where $N_{\geq \frac{1}{2}}$ is the number of steps of length larger than $\frac{1}{2}$ among the small steps of a phase. Because $N_{\text{small}}$, the number of small steps in one phase, follows a geometric distribution of parameter $\alpha^{-1}$, we have $N_{\text{small}} = \Omega(\alpha^{-1})$ with constant probability. Furthermore, as a small step has length at least $\frac{1}{2}$ with constant probability, we have that

$$N_{\geq \frac{1}{2}} = \Theta(N_{\text{small}}) = \Omega(\alpha^{-1}),$$

with constant probability. Indeed, $N_{\geq \frac{1}{2}}$ follows a binomial distribution, and we are using the median property of such distributions.

By choosing $c_1$ such that $T^\star = c_1D^{\mu-1}$ is small enough, since $\alpha^{-1} = \Theta(D^{\mu-1})$, we have $\Pr(N_{\geq \frac{1}{2}} \geq T^\star) \geq c'$ for some constant $0 < c' < 1$. This implies that for some constant $0 < c'' < 1$,

$$\mathbb{E}(N_{\text{long-phases}}^R) \geq c''R.$$  

Hence,

$$\mathbb{E}(N_{\text{short-phases}}^R) \leq (1 - c'')R,$$

where $N_{\text{short-phases}}^R$ is the number of short (i.e., non-long) phases. By Markov’s inequality, for any $c_2 > 0$, we have $\Pr(N_{\text{short-phases}}^R \geq c_2R) \leq \frac{1 - c''}{c_2}$, which is a positive, strictly less than 1, constant, by a suitable choice of $c_2$. For this choice, we have

$$\Pr(N_{\text{long-phases}}^R \geq (1 - c_2)R) = \Pr(N_{\text{short-phases}}^R < c_2R) \geq 1 - \frac{1 - c''}{c_2} = \Omega(1).$$

Returning to Eq. (19), we get that with constant probability

$$T^R = \Omega(T^\star R) = \Omega(RD^{\mu-1}),$$

which proves Lemma 17. □

We conclude by using Lemmas 16 and 17. Specifically, for the constants $c > 0$ and $0 < q < 1$ of Lemma 17 and the constant $\delta = 1 - q/2$ in Lemma 16 for some choice of the constant $\tilde{c} > 0$ in the definition of $R$, we obtain:

- $\Pr(T^R \geq cRD^{\mu-1}) > q$, and
• \( \Pr(N^R_{\text{discs}} < M/2) > \delta \).

Using a union bound argument, this implies that with probability at least \( q+\delta - 1 = q/2 \), we have both \( N^R_{\text{discs}} < M/2 \) and \( T^R = \Omega(RD^{\mu-1}) \). Hence, with constant probability, the searcher takes time \( \Omega(RD^{\mu-1}) \) to find the target. Therefore, the expected time needed to find the target is

\[
\tau = \left\{ \begin{array}{ll}
\Omega(MD^{\mu-1}) = \Omega(nD^{\mu-3}) & \text{if } 2 < \mu < 3, \\
\Omega(MD^{3-1}) = \Omega\left(\frac{n}{\log D}\right) & \text{if } \mu = 3,
\end{array} \right.
\]

where we used the definition of \( R \) in Eq. (18) and the fact that \( M = \Theta(n/D^2) \). Dividing by the optimal time \( \Theta(n/D) \), we get

\[
\frac{\tau}{\Theta(n/D)} = \left\{ \begin{array}{ll}
\Omega(D^\varepsilon) & \text{if } \mu = 2 + \varepsilon, \text{ where } 0 < \varepsilon < 1, \\
\Omega\left(\frac{D}{\log D}\right) & \text{if } \mu = 3,
\end{array} \right.
\]

as desired. This completes the proof of Theorem 12 in the continuous detection model.

\[\square\]

C Scale-sensitivity of the intermittent Cauchy Walk

We take \( n > 4 \) for technical reasons, and let \( \ell_{\text{max}} = \sqrt{n}/2 \). As stated in the previous section, the overrun of the intermittent Cauchy walk \( X^{\text{cauchy}} \) for a target of diameter \( D \) on the torus is \( \Omega(\log n) \). The goal of this section is to prove the following theorem which states that this lower bound is nearly matched.

**Theorem 18.** Consider the Cauchy walk process \( X^{\text{cauchy}} \) on the torus \( \mathbb{T}_n \). The hitting time of \( X^{\text{cauchy}} \) with respect to a target \( S \) of diameter \( D \) is

\[
t^X_{\text{detect}}(S) = O\left(\frac{n \log^3 n}{D}\right).
\]

Consequently, the overrun of \( X^{\text{cauchy}} \) for a target of diameter \( D \) is \( O(\log^3 D) \).

Theorem 18 concerns the Cauchy walk on the two-dimensional torus. As the one-dimensional Cauchy walk is fairly well understood, it is tempting to analyze the two-dimensional walk by projecting it on the two axes and using the properties of the one-dimensional walk on these projections. However, this approach needs to somehow handle the fact that these projections are not independent of each other. As we could not find an easy way to overcome this dependence issue, we prove Theorem 18 following a different line of arguments, that directly examine the two-dimensional process.

To prove Theorem 18, we can assume without loss of generality that the process starts at the origin, i.e., that \( X^{\text{cauchy}}(0) = 0 \).

Claim 3 implies that in order to find the detecting time \( t^X_{\text{detect}}(S) \) of \( S \), it is sufficient to identify the expected number of steps until detecting \( S \), as

\[
t^X_{\text{detect}}(S) = \mathbb{E}(m^X_{\text{detect}}(S)) \cdot \tau = \Theta(\mathbb{E}(m^X_{\text{detect}}(S)) \cdot \log n).
\]
Now let $Z$ be the process on $\mathbb{R}^2$ that evolves with the same steps $V(s)$ as $X_{\text{cauchy}}$, i.e. $Z(m) = \sum_{s=1}^{m} V(s)$. Note that the projection of $Z$ on the torus $[-\sqrt{n}/2, \sqrt{n}/2]^2 \subset \mathbb{R}^2$ is $X_{\text{cauchy}}$.

The next lemma establishes a connection between $\mathbb{E}(m_{\text{detect}}^X(S))$ and the process $Z$ on $\mathbb{R}^2$. Given a set $S$, recall that $B(S)$ is the set of points at distance at most 1 from $S$, and that $Z(m)$ detects $S$ if and only if $Z(m) \in B(S)$.

**Lemma 19.** Consider a random walk process $Z$ on $\mathbb{R}^2$ and its projection $X$ on the torus $\mathbb{T}_n$ and denote by $Z^{z_0}$ the process $Z$ starting at $Z(0) = z_0$. Let $S \subset \mathbb{T}_n$. For any $m_0$, 

$$\mathbb{E}(m_{\text{detect}}^X(S)) = O \left( m_0 \cdot \frac{\sup_{z_0 \in B(S)} \sum_{m_0}^{m} \Pr(Z^{z_0}(m) \in B(S))}{\sum_{m=m_0}^{2m_0} \Pr(Z(m) \in B(S))} \right).$$

(21)

We provide a formal proof of Lemma 19 in Section C.1. The proof is based on the technique in Adler et al. [42] relying on the identity $\Pr(N \geq 1) = \frac{\mathbb{E}(N)}{\mathbb{E}(N | N \geq 1)}$, that holds for any non-negative random variable $N$.

Lemma 19 allows to deduce Theorem 18 from pointwise bounds on the Cauchy process $Z$ on $\mathbb{R}^2$, defined by Eq. (1). The next lemma provides a lower bound on the p.d.f $p_Z(m)$, of the process at step $m$.

**Lemma 20.** For any constant $\alpha > 0$, there exists a constant $c > 0$ such that for any integer $m \in [1, \alpha \ell_{\text{max}}]$, and any $x \in \mathbb{R}^2$, with $\|x\| \leq m$, 

$$p_Z(m)(x) \geq \frac{c}{m^2}.$$ 

From Lemma 20, we immediately deduce that the probability that $Z(m)$ detects a point $x \in \mathbb{R}^2$ is $\Omega(\int_{y \in B(x)} cm^{-2} dy) = \Omega(cm^{-2})$, where $B(x) = B(\{x\})$. This lower bound is complemented by the following upper bound.

**Lemma 21.** For any constant $\alpha > 0$, there exists a constant $c' > 0$ such that, for any integer $m \in [2, \alpha \ell_{\text{max}}]$ and any $x \in \mathbb{R}^2$, we have 

$$\Pr(Z(m) \in B(x)) \leq \frac{c' \log^2 m}{m^2}.$$ 

Lemmas 20 and 21 are formally proved in Section C.2. Let us give here a sketch of the proofs. Using the monotonicity property, the lower bound stated in Lemma 20 follows once we prove that with at least some constant probability, the process at step $m$ belongs to the ring $\{x \mid \|x\| \in [m, cm]\}$ for some constant $c > 1$. This is because the area of this ring is roughly $m^2$, and each point in it is further from 0 than $x$, and hence, by monotonicity, less likely to be visited at step $m$. In order to establish the lower bound on the probability to be in the ring at step $m$, we first prove that with some constant probability, at some step before $m$, the walk goes to a distance at least $2m$.

Next, conditioning on that event, we prove that with a constant probability, the walk does not get much further away, i.e., it stays at a distance of at least $m$. To prove the latter claim, we use Chebyshev’s inequality. It implies, for a one-dimensional process, that the distance traveled in $m$
steps is governed by $\sqrt{m}$ times the standard deviation of the step-length process. Here the standard deviation is too large (roughly $\sqrt{n}$), however, we can reduce it by conditioning on the event that none of the $m$ step-lengths are significantly larger than $m$, which occurs with a constant probability. Finally, we prove that by taking a sufficiently large constant $c$, it can be guaranteed that with a large constant probability, the walk at step $m$ is at most at distance $cm$. Making sure that all these constant probability events happen simultaneously, we then establish the desired constant lower bound on the probability to be in the aforementioned ring at step $m$.

For the proof of the upper bound in Lemma 20, we first show that because of the monotonicity property, it is sufficient to prove that the probability to detect 0 at step $m$ is small, i.e., that

$$\Pr(Z(m) \in B(0)) = O\left(\frac{\log^2 m}{m^2}\right).$$

Intuitively, to establish this, we first argue that with high probability in $m$, at some step before step $m$, the process has gone to a distance $d = \Omega(\frac{m}{\log m})$. By Corollary 6, the probability density function at any point in $B(0)$ would then be at most $O(\frac{1}{\sqrt{m^2}})$, which is the desired bound.

**Proof of Theorem 18 assuming the aforementioned Lemmas.** Given the connected set $S$ of diameter $D \geq 1$, we first construct a subset $S'$, containing $\Theta(D)$ isolated points of $S$ that stretch over distance of roughly $D$, as follows. Take two points $u = (u_1, u_2)$ and $v = (v_1, v_2)$ in $S$ that are at distance $D$ from each other, so that $\max\{|u_1 - v_1|, |u_2 - v_2|\} \geq D/2$. Let us assume, without loss of generality, that $v_1 - u_1 \geq D/2$. Since $S$ is connected, for every $z \in [u_1, v_1]$, there exists $\phi(z)$ such that $(z, \phi(z)) \in S$. Let $d = |v_1 - u_1| = \Theta(D)$. For integer $i \in \{0, 1, \ldots, |d|\}$, define

$$s(i) = (u_1 + i, \phi(u_1 + i)),$$

and let $S' = \{s(i) \mid i = 0, 1, \ldots, |d|\}$. Note that $|S'| = \Theta(D)$. Since $S' \subseteq S$, an upper bound on the detecting time of $S'$ is an upper bound on the detecting time of $S$. It is therefore sufficient to restrict attention to $S'$ and upper bound its detecting time. For that purpose we need to bound the time until visiting a point in $B(S')$, the set of points of distance at most 1 from $S'$. Note that the area of $B(S')$ is $|B(S')| = \Omega(D)$. We also remark, that although $B(S')$ may not be connected, it may help the reader to imagine $B(S')$ as a horizontal cylinder of length $\Theta(D)$ and radius 1, i.e., to consider that $\phi(u_1 + i)$ does not depend on $i$. Indeed, we will not require any condition on the $y$-coordinates of the $s(i)$’s.

In order to upper bound $\mathbb{E}(m_{\text{detect}}^X(B(S'))) = \sum_{m=m_0}^{2m_0} \Pr(Z(m) \in B(S'))$ we shall apply Lemma 19 with $m_0 = \sqrt{m}$. Note that $2m_0 \leq \alpha \ell_{\text{max}}$ for $\alpha = 4$. We shall furthermore lower bound the denominator in the r.h.s of Eq. (21) and upper bound the numerator. Both these terms concern the Cauchy process $Z$ with cut off $\ell_{\text{max}}$ on $\mathbb{R}^2$.

Let us begin with the lower bound. With this setting of $m_0$, any $x \in B(S') \subseteq B(\mathbb{T}_n) \subseteq [-\sqrt{m}/2 - 1, \sqrt{m}/2 + 1]^2$ trivially satisfies $\|x\| \leq m$, for any $m \geq m_0 + 1$, and we can apply Lemma 20 to get a lower bound on the denominator in the r.h.s of Eq. (21):

$$\sum_{m=m_0+1}^{2m_0} \Pr(Z(m) \in B(S')) = \sum_{m=m_0+1}^{2m_0} \int_{x \in B(S')} p_m^Z(x)dx \geq \sum_{m=m_0+1}^{2m_0} \frac{c}{m^2} |B(S')| = \Omega\left(\frac{D}{\sqrt{m}}\right).$$
Next, we provide an upper bound to the numerator of the r.h.s of Eq. (21) which is the number of returns to $S'$ conditioning on the fact that $Z(0) = z$, for some $z \in B(S')$. Let us denote this process by $Z^z$ (note that $Z = Z^0$). Then,

$$\sum_{m=0}^{m_0} \Pr(Z^z(m) \in B(S')) \leq 2 + \sum_{m=2}^{m_0} \Pr(Z^z(m) \in B(S')).$$

(22)

Clearly, the probability density function $p^{Z^z(m)}$ of $Z^z(m)$ is obtained by a translation from $p^{Z(m)}$. Thus, by Corollary 5 we have for any $y \in \mathbb{R}^2$:

$$p^{Z^z(m)}(y) \leq \frac{1}{\|y - z\|^2}.$$ 

In particular, for $y$ such that $\|y - z\| \geq 2$,

$$\Pr(Z^z(m) \in B(y)) \leq \frac{1}{(\|y - z\| - 1)^2},$$

(23)

since every $w \in B(y)$ satisfies $\|w - z\| \geq \|y - z\| - 1 \geq 0$.

Next, as $z \in B(S')$, consider an index $i_z \in \{0, \ldots, d - 1\}$ for which $z \in B(s(i_z))$. Let $r_m = \frac{m}{\sqrt{c\log m}}$ with $c$ being the constant $c'$ mentioned in Lemma 21. To exploit Eq. (23), we define

$$I = \{i \in \{0, \ldots, d - 1\} \mid |s(i)_1 - s(i_z)_1| = |i - i_z| \leq r_m + 2\},$$

and $I^c = \{0, \ldots, d - 1\} \setminus I$. We proceed with the following decomposition:

$$\Pr(Z^z(m) \in B(S')) \leq \sum_{i \in I} \Pr(Z^z(m) \in B(s(i))) + \sum_{i \in I^c} \Pr(Z^z(m) \in B(s(i))).$$

(24)

By construction, $|I| \leq 2(r_m + 2) + 1$. Hence, using Lemma 21 the first sum in the r.h.s of Eq. (24) is at most:

$$\sum_{i \in I} \Pr(Z^z(m) \in B(s(i))) \leq \frac{|I|}{r_m^2} = O\left(\frac{1}{r_m}\right).$$

Next, we aim to upper bound the sum on $I^c$. By the triangle inequality, for any $i \in I^c$, we have $\|s(i) - z\| \geq \|s(i) - s(i_z)\| - 1 \geq |i - i_z| - 1 > 1$. Hence, by Eq. (23), we get:

$$\sum_{i \in I^c} \Pr(Z^z(m) \in B(s(i))) \leq \sum_{i \in I^c} \frac{1}{\|s(i) - z\|^2} \leq \sum_{i \in I^c} \frac{1}{(|i - i_z| - 2)^2} \leq \sum_{k \in \mathbb{Z}, |k| \geq [r_m]} \frac{1}{k^2} = O\left(\frac{1}{r_m}\right),$$

where we used in the last line that $i \in I^c \subset \{i_z + k \mid k \in \mathbb{Z} \text{ and } |k| > r_m + 2\}$. Thus, we get by Eq. (24):

$$\Pr(Z^z(m) \in B(S')) = O\left(\frac{1}{r_m}\right).$$
Plugging this in Eq. (22), together with the definition \( r_m = \frac{m}{\sqrt{\epsilon \log m}} \), and the fact that \( m_0 = O(\sqrt{n}) \), we get:

\[
\sum_{m=0}^{m_0} \Pr(Z^z(m) \in B(S')) = 2 + O\left( \sum_{m=2}^{m_0} \frac{\log m}{m} \right) = O(\log^2 n),
\]

which stands for any \( z \in B(S') \). Altogether, the fraction in Eq. (21) satisfies:

\[
\frac{\sup_{z \in B(S')} \sum_{m=0}^{m_0} \Pr(Z^z(m) \in B(S'))}{\sum_{m=m_0}^{2m_0} \Pr(Z(m) \in B(S'))} = O\left( \frac{\sqrt{n}}{D} \cdot \log^2 n \right).
\]

Together with the fact that \( m_0 = O(\sqrt{n}) \), Lemma 19 implies that \( \mathbb{E}(m^X_{\text{detect}}(S)) = O\left(\frac{n}{D} \log^2 n \right) \).

Finally, using Claim 3 and the fact that \( \tau = \Theta(\log n) \), we have

\[
\tau^X_{\text{detect}}(S) = O\left( \frac{n \log^3 n}{D} \right),
\]

and since this is true for any connected set \( S \subseteq \mathbb{T}_n \) of diameter \( D \), we obtain \( \tau^X_{\text{detect}}(n, D) = O\left( \frac{n \log^3 n}{D} \right) \), as desired. \( \square \)

C.1 Proof of Lemma 19

The goal of this section is to prove of Lemma 19. Recall, we consider a random walk process \( Z \) on \( \mathbb{R}^2 \) and its projection \( X \) on the torus \( \mathbb{T}_n \). Let \( S \subseteq \mathbb{T}_n \). Our goal is to show that for any \( m_0 \),

\[
\mathbb{E}(m^X_{\text{detect}}(n, D)) = O\left( m_0 \frac{\sup_{z_0 \in B(S)} \sum_{m=0}^{m_0} \Pr(Z^{z_0}(m) \in B(S))}{\sum_{m=m_0}^{2m_0} \Pr(Z(m) \in B(S))} \right). \tag{25}
\]

**Proof.** We begin with the following claim that shows that if the probability to detect \( S \) by step \( m \) is at least \( p \) for any starting point, then the expected detecting step is at most \( m/p \). The claim will then be used to prove the lemma by showing that the inverse of the supremum in Eq. (21) is a lower bound for \( \Pr(m^X_{\text{detect}}(S) \leq 2m_0) \).

**Claim 22.** Fix an integer \( m > 0 \) and a real number \( q > 0 \) and a set \( S \subseteq \mathbb{T}_n \). Denote by \( X^x \) the process \( X \) starting at \( X(0) = x \). If, for any \( x \in \mathbb{T}_n \), we have \( \Pr(m^X_{\text{detect}}(S) \leq m) \geq q \) then \( \mathbb{E}(m^X_{\text{detect}}(S)) \leq mq^{-1} \).

**Proof of Claim 22.** The proof of the claim is simple. Given a set \( S \), define a Bernoulli variable \( \chi \) as follows. Consider \( m \) steps of the process and define \( \chi \) to be “success” if and only if the process hits \( S \) within these \( m \) steps. Note that \( \chi \) has probability at least \( q \) to be “success” regardless of where the process starts, by hypothesis. Hence, the expected number of trials until \( \chi \) succeeds is at most \( 1/q \). This translates to \( \mathbb{E}(m^X_{\text{detect}}(S)) \leq mq^{-1} \), and establishes Claim 22. \( \square \)

To conclude the proof of Lemma 19 relying on Claim 22 it is sufficient to prove that, for any \( S \subseteq \mathbb{T}_n \),

\[
\Pr(m^X_{\text{detect}}(S) \leq 2m_0) \geq \frac{\sum_{m=m_0}^{2m_0} \Pr(Z(m) \in B(S))}{\sup_{z_0 \in B(S)} \sum_{m=0}^{m_0} \Pr(Z^{z_0}(m) \in B(S))}. \tag{26}
\]
For this, we rely on the following identity (see also [42, 43]). If $N$ is a non-negative random variable then:

$$\Pr(N \geq 1) = \frac{\mathbb{E}(N)}{\mathbb{E}(N \mid N \geq 1)}. \quad (27)$$

We employ this identity for the random variable $N_S(m_0, 2m_0)$ which is the number of times $Z$ visits $B(S)$ between steps $m_0$ and $2m_0$ included. Note that this quantity is positive if and only if $B(S)$ is visited during this interval by $Z$. Moreover, since $S \subset T_n$ and $X$ is the projection of $Z$ on the torus, then $Z(m) \in B(S)$ implies that also $X(m) \in B(S)$. Therefore,

$$\Pr(m^X_{detect}(S) \leq 2m_0) \geq \Pr(N_S(m_0, 2m_0) \geq 1). \quad (28)$$

Note that $N_S(m_0, 2m_0) = \sum_{m=m_0}^{2m_0} 1_{Z(m) \in B(S)}$, so that

$$\mathbb{E}(N_S(m_0, 2m_0)) = \sum_{m=m_0}^{2m_0} \Pr(Z(m) \in B(S)). \quad (29)$$

Note also that the denominator in Eq. (27) applied to $N_S(m_0, 2m_0)$ verifies

$$\mathbb{E}(N_S(m_0, 2m_0) \mid N_S(m_0, 2m_0) \geq 1) = \mathbb{E}(N_S(m_0, 2m_0) \mid Z(m) \in B(S) \text{ for some } m \in [m_0, 2m_0]) \leq \sup_{Z_0 \in B(S)} \mathbb{E}(N_S(m_0, 2m_0) \mid Z(m_0) = z_0)
\leq \sup_{Z_0 \in B(S)} \mathbb{E}(N_S(0, m_0) \mid Z(0) = z_0),$$

where the first inequality comes from the fact that visiting $B(S)$ earlier (i.e., for $m = m_0$ instead of $m > m_0$) can only increase the number of returns to $B(S)$, and the second inequality is a consequence of the Markov property. Finally, write, as above,

$$\sup_{Z_0 \in B(S)} \mathbb{E}(N_S(0, m_0) \mid Z(0) = z_0) = \sup_{Z_0 \in B(S)} \sum_{m=0}^{m_0} \Pr(Z^{z_0}(m) \in S). \quad (30)$$

Therefore, when applied to $N_S(m_0, 2m_0)$, Eq. (27), combined with Eqs. (28), (29) and (30), implies that

$$\Pr_0(m^X_{detect}(S) \leq 2m_0) \geq \frac{\sum_{m=m_0}^{2m_0} \Pr(Z(m) \in B(S))}{\sup_{Z_0 \in B(S)} \sum_{m=0}^{m_0} \Pr(Z^{z_0}(m) \in B(S))}.$$

This establishes Eq. (26), and thus completes the proof of Lemma 19.

C.2 Proofs of Lemmas 20 and 21

In this section we aim to prove the following lower and upper bounds, stated in Lemmas 20 and 21 respectively. The proof of Lemma 20 is given in Section C.2.2 and the proof of Lemma 21 is given in Section C.2.3. Before presenting these proofs, let us first establish lower and upper bounds on the distance traveled by the walk at step $m$. \qed
C.2.1 Superdiffusive properties of the Cauchy walk on $\mathbb{R}^2$

We first remark that the probability to choose a length in a given interval is easily computed from Eq. (1).

Observation 23. The probability to do a step of length at most $\ell > 0$ is $a\ell$ if $\ell \leq 1$ and $a(2 - \frac{1}{\ell})$ if $\ell > 1$. For integers $\ell_{\text{max}} \geq \ell_2 \geq \ell_1 \geq 1$, the probability to choose a length in $[\ell_1, \ell_2]$ is $a(\frac{1}{\ell_1} - \frac{1}{\ell_2})$.

The next claim quantifies the probability that the Cauchy process goes to a distance of at least $d$ after $m$ steps. In particular, it shows that in step $m$, the process is at a distance of $\Omega(\sqrt{m})$ with constant probability, and that it is at a distance of $\Omega(m/\log m)$ with high probability in $m$.

Claim 24. For any integer $m \geq 2$ and any real $d \in [1, \frac{\ell_{\text{max}}}{3}]$ we have,
$$\Pr(\exists s \leq m \text{ s.t. } \|Z(s)\| \geq d) \geq 1 - e^{-cm/d},$$
for some constant $c > 0$. In particular this lower bound is at least

- $1 - O(m^{-2})$ if $d = c' \frac{m}{\log m}$ with $c' > 0$ a small enough constant,
- $\Omega(1)$ if $d = c'm$ for any constant $c' > 0$ with $c'm \leq \ell_{\text{max}}/3$.

Proof. By Observation 23, the probability that a given step has a length at least $2d$ is $a(\frac{1}{2d} - \frac{1}{\ell_{\text{max}}}) \geq \frac{a}{6d}$. Since the steps are independent, the probability of the event $\mathcal{A}$ that at least one of the steps $1, \ldots, m$ has a length at least $2d$ is
$$\Pr(\mathcal{A}) \geq 1 - \left(1 - \frac{a}{6d}\right)^m.$$ Writing $(1 - a/6d)^m = e^{m \log(1 - a/6d)} \leq e^{-cm/d}$, for some constant $c > 0$, we get
$$\Pr(\mathcal{A}) \geq 1 - e^{-cm/d}.$$ To conclude, it suffices to show that $\mathcal{A}$ implies that there exists a step $s \leq m$ for which $\|Z(s)\| \geq d$. Indeed, suppose that $\mathcal{A}$ occurs and let $s \leq m$ be the first step of length $2d$ or more. Then,

- Either $\|Z(s - 1)\| \geq d$, in which case we are done.
- Or $\|Z(s - 1)\| < d$. In this case, as $Z(s) = Z(s - 1) + V(s)$, we have $\|Z(s)\| \geq \|V(s)\| - \|Z(s - 1)\| > 2d - d = d$.

This concludes the proof of Claim 24. □

Claim 24 asserts that, with some probability, the walk goes far from 0. Conversely, the next claim says that with some constant probability, the walk does not get too far.

Claim 25. For any constant $c > 0$, there exists a constant $\delta > 0$ such that, for any two integers $1 \leq s \leq m$, we have $\Pr(\|Z(s)\| \leq cm) \geq \delta$. 

For any constant $0 < \delta < 1$, there exists a (large enough) constant $c > 0$ such that, for any two integers $1 \leq s \leq m$, we have $\Pr(\|Z(s)\| \leq cm) \geq \delta$.

Proof. Fix an integer $m \geq 1$ and let $c''$ be a constant, to be chosen later. Let $\mathcal{A}$ denote the event that each of the first $m$ steps has length at most $\ell = c''m$. We have, for any integer $s \leq m$, and any constant $c > 0$,

$$\Pr(\|Z(s)\| \leq cm) \geq \Pr(\mathcal{A}) \cdot \Pr(\|Z(s)\| \leq cm \mid \mathcal{A}). \quad (31)$$

We shall study separately each term in the r.h.s of Eq. (31), and establish the following:

- For the first item of Claim 25, we shall take $c'' > 0$ so that both factors are constants (hence their multiplication is at least some constant $\delta$),
- For the second item of Claim 25, where the bound $\delta$ is given, we will show that both terms can be made at least $\sqrt{\delta}$ by choosing $c$ and $c''$ appropriately.

Proceeding with the first term in the r.h.s of Eq. (31), by Observation 23, we have:

$$\Pr(\mathcal{A}) = \begin{cases} (ac''m)^m & \text{if } c''m \leq 1 \\ (2a)^m(1 - \frac{1}{2c''m})^m & \text{if } c''m \in [1, \ell_{\max}] \\ 1 & \text{if } c''m \geq \ell_{\max} \end{cases}.$$ 

For $1 \leq m \leq \frac{1}{\alpha}$, we have $(ac''m)^m \geq (ac''m)^{\frac{1}{\alpha}}$ as $ac''m \leq c''m \leq 1$, and $(ac''m)^{\frac{1}{\alpha}} \geq (ac'')^{\frac{1}{\alpha}}$ as $m \geq 1$. For the second item, note that the function $(1 - \frac{a}{x})^x = e^{x\log(1 - \frac{a}{x})}$ is increasing in $x \geq \alpha$ and thus, for $x \geq 2\alpha$, we have $(1 - \frac{a}{x})^x \geq 2^{-2\alpha}$. Applying this with $\alpha = \frac{1}{2c''}$, we have, $(1 - \frac{1}{2c''m})^m \geq 2^{-\frac{1}{c''}}$, for $m \geq \frac{1}{c''}$. Overall, using $2a \geq 1$, we get

$$\Pr(\mathcal{A}) \geq \begin{cases} (\frac{c''}{2})^{\frac{1}{\alpha}} & \text{if } c''m \leq 1 \\ 2^{-\frac{1}{c''}} & \text{if } c''m \in [1, \ell_{\max}] \\ 1 & \text{if } c''m \geq \ell_{\max} \end{cases}.$$ 

Hence,

- $\Pr(\mathcal{A}) = \Omega(1)$ for any given $c'' > 0$.

- Furthermore, with respect to the second item of Claim 25 where $0 < \delta < 1$ is given, we can choose $c''$ large enough (in particular, we take $c'' \geq 1$ so that $c''m \geq 1$), to ensure that $\Pr(\mathcal{A}) \geq 2^{-\frac{1}{c''}} \geq \sqrt{\delta}$.

We are now ready to lower bound the second factor in Eq. (31), namely, $\Pr(\|Z(s)\| \leq cm \mid \mathcal{A})$.

We begin with a notation: If $X$ is a random variable, let us write $X^\mathcal{A}$ for the random variable $X$ conditioned on the occurrence of $\mathcal{A}$. Our first goal is to prove that

$$\Pr(\|Z^\mathcal{A}(s)\| \leq cm) \geq 1 - \frac{8sE(\|V\|^2)}{c^2m^2}, \quad (32)$$
We then conclude the proof of Claim 25 by observing the following. Eq. (32) will be established by applying Chebyshev’s inequality on each of the projections on the axes and using a union bound argument. Specifically, decomposing the walk $Z$ on the two axes, by writing $Z = (Z_1, Z_2)$, we first use a union bound to obtain:

\[
\Pr(\|Z^A(s)\| > cm) \leq \Pr(\exists i = 1, 2 \text{ s.t. } |Z_i^A(s)| > cm/2) \\
\leq \Pr(|Z^A(s)| > cm/2) + \Pr(|Z_2^A(s)| > cm/2) \\
\leq 2\Pr(|Z_1^A(s)| > cm/2),
\]

where we used the symmetry to deduce that $Z_1$ and $Z_2$ share the same distribution. Hence,

\[
\Pr(\|Z^A(s)\| \leq cm) \geq 1 - 2\Pr(|Z_1^A(s)| > cm/2).
\]

Next, we aim to lower bound the r.h.s. Relying on the fact that the expectation of $Z_1^A(s)$ is 0 for any $s$, by Chebyshev’s inequality, we have:

\[
\Pr(|Z_1^A(s)| > cm/2) \leq \frac{4\text{Var}(Z_1^A(s))}{c^2m^2}.
\]

Since $Z_1^A(s)$ is the sum of $s$ independent steps that follow the same law as $V_1^B$, we have:

\[
\text{Var}(Z_1^A(s)) = s\text{Var}(V_1^B).
\]

As the expectation of $V_1^B$ is zero, we have $\text{Var}(V_1^B) = \mathbb{E}((V_1^B)^2)$. Furthermore, since $|V_1^B| \leq \|V^B\|$, we obtain:

\[
\text{Var}(Z_1^A(s)) \leq s\mathbb{E}(\|V^B\|^2),
\]

which concludes the proof of Eq. (32). Next, let us estimate $\mathbb{E}(\|V^B\|^2)$. If, on the one hand, $c''m \leq 1$, then, when conditioning on $A$, the length of a step is chosen uniformly at random in $[0, c''m]$. Thus, its second moment is

\[
\mathbb{E}(\|V^B\|^2) = \int_0^{c''m} \ell^2 \frac{d\ell}{c''m} = \frac{(c''m)^2}{3}. \tag{33}
\]

On the other hand, if $c''m \geq 1$, then $V^B$ is a Cauchy walk with cut off $\ell_{\max} = c''m$. Hence, its second moment is

\[
\mathbb{E}(\|V^B\|^2) = a'\int_0^1 \ell^2d\ell + a'\int_1^{c''m} \ell^2\ell^{-2}d\ell \\
\leq a'\int_0^{c''m} 1d\ell = a'c''m \leq c''m. \tag{34}
\]

Overall, by Eqs. (32), (33) and (34) we find that, for $s \leq m$,

\[
\Pr(\|Z^A(s)\| \leq cm) \geq \begin{cases}
1 - \frac{8sc''}{3c''m} & \text{if } c''m \leq 1 \\
1 - \frac{8sc''}{c^2m} & \text{if } c''m \geq 1
\end{cases}
\]

\[
\geq \begin{cases}
1 - \frac{8sc''}{3c''m} & \text{if } c''m \leq 1 \\
1 - \frac{8sc''}{c^2m} & \text{if } c''m \geq 1
\end{cases}.
\]

We then conclude the proof of Claim 25 by observing the following.
• For the first item of Claim 25, we have proved that $\Pr(A) = \Omega(1)$ for any constant $c'' > 0$. Hence, we may now choose $c''$ small enough so that $\Pr(\|Z^A(s)\| \leq cm) = \Omega(1)$.

• For the second item of Claim 25, we have already chosen $c''$ to be large (in order to have $\Pr(A) \geq \sqrt{\delta}$, but we are free to choose $c$ large enough so that $\Pr(\|Z^A(s)\| \leq cm) \geq \sqrt{\delta}$.

\[\square\]

C.2.2 Proof of Lemma 20 (lower bound)

In this section we prove the following:

**Lemma 20 (restated).** For any constant $\alpha > 0$, there exists a constant $c > 0$ such that for any integer $m \in [1, \alpha \ell_{\text{max}}]$, and any $x \in \mathbb{R}^2$, with $\|x\| \leq m$,

$$p^{Z(m)}(x) \geq \frac{c}{m^2}.$$  

Proof. First note that for $m = 1$, the lemma holds by the definition of the Lévy process. Let us therefore consider an integer $m \geq 2$.

By the monotonicity property (Corollary 5), it is enough to prove that there is some constant $c' > 1$ such that,

$$\Pr(m \leq \|Z(m)\| \leq c'm) = \Omega(1). \tag{35}$$

Indeed, if this holds, then, since the area of the ring $\{y \in \mathbb{R}^2 \text{ s.t. } m \leq \|y\| \leq c'm\}$ is $\Theta(m^2)$, then we would have that for at least one point $u$ in this ring, $p^{Z(m)}(u) = \Omega(m^{-2})$. Then, by monotonicity, for $x \in \mathbb{R}^2$ such that $\|x\| \leq m$, we would have $p^{Z(m)}(x) \geq p^{Z(m)}(u) = \Omega(m^{-2})$ which is the desired lower bound.

We thus proceed to prove Eq. (35). For this, let us define, for a given $m \in [2, \alpha \ell_{\text{max}}]$, the event

$$A_{\text{far}} = \exists s \leq m \text{ s.t. } \|Z(s)\| \geq 2m.$$  

We next prove the following claim.

**Claim 26.** $\Pr(A_{\text{far}}) = \Omega(1)$, where the constant in lower bound does not depend on $m$.

Proof of Claim 26. By Claim 24, we immediately get that the claim holds for any $m \in [2, \ell_{\text{max}}/6]$. We next show that the claim holds also for $m \in [\ell_{\text{max}}/6, \alpha \ell_{\text{max}}]$. Intuitively, we prove this using a constant number of iterations. Each iteration consists of at most $m' = \alpha' \ell_{\text{max}}$ steps, with $\alpha'$ a small constant, during which we are guaranteed to go a distance of $\ell_{\text{max}}/3$ with constant probability. Because the direction is chosen uniformly at random, at the cost of reducing this probability by a constant factor, we can further impose that the $x$-coordinate increases by a factor of, say, $\ell_{\text{max}}/5$. As these iterations are independent, and since $\alpha$ is a constant, we can guarantee that up to step $m = \alpha \ell_{\text{max}}$, the process goes away to a distance of at least $2\alpha \ell_{\text{max}}$ with constant probability.
Formally, first notice that we can take \( \alpha > 1 \) without loss of generality. Note now that since \( m \in [\ell_{\text{max}}/6, \alpha \ell_{\text{max}}] \), the second item in Claim 24 implies that:

\[
\Pr\left( \exists s \leq \frac{m}{10\alpha} \text{ s.t. } \|Z(s)\| \geq \frac{\ell_{\text{max}}}{3} \right) \geq c'_{\alpha},
\]

for some constant \( c'_{\alpha} > 0 \). As a consequence, since the direction of \( Z(s) \) is distributed uniformly at random, we have:

\[
\Pr\left( \exists s \leq \frac{m}{10\alpha}, Z_1(s) \geq \frac{\ell_{\text{max}}}{4} \right) \geq c_{\alpha},
\] (36)

for some constant \( c_{\alpha} > 0 \). When this occurs, let \( s_1 \leq \frac{m}{10\alpha} \) be such that \( Z_1(s_1) \geq \frac{\ell_{\text{max}}}{4} \). By the Markov property, starting from step \( s_1 \), we can then apply again (36) to show that with probability \( c_{\alpha} \), there is a \( s_2 \leq s_1 + \frac{m}{10\alpha} \leq 2 \frac{m}{10\alpha} \) such that \( Z_1(s_2) \geq Z_1(s_1) + \frac{\ell_{\text{max}}}{4} \geq 2 \frac{\ell_{\text{max}}}{4} \). Overall, this happens with probability \( c_{\alpha}^2 \). Repeating this \( \lceil 9\alpha \rceil \) times, we finally get:

\[
\Pr\left( \exists s \leq \lceil 9\alpha \rceil \frac{m}{10\alpha}, Z_1(s) \geq \lceil 9\alpha \rceil \frac{\ell_{\text{max}}}{4} \right) \geq c_{\alpha}^{\lceil 9\alpha \rceil},
\]

which is a positive constant. Because \( \alpha > 1 \), this implies \( \Pr(\exists s \leq m, Z_1(s) \geq 2\alpha \ell_{\text{max}}) = \Omega(1) \). As \( 2\alpha \ell_{\text{max}} \geq 2m \) and \( \|Z\|(s) \geq |Z_1(s)| \), this, in turn, implies \( \Pr(A_{\text{far}}) = \Omega(1) \), completing the proof of Claim 26.

Next, conditioning on \( A_{\text{far}} \), we write:

\[
\Pr(\|Z(m)\| \geq m \mid A_{\text{far}}) \geq \min_{s \leq m} \Pr(\|Z(m)\| \geq m \mid \|Z(s)\| \geq 2m)
\]

\[
\geq \min_{s \leq m} \Pr(\|Z(m - s)\| \leq m),
\]

where we used the Markov property, and the spatial homogeneity of the process, in the latter inequality. In words, in the r.h.s. of Inequality (37), we examine the probability to be at a high distance (i.e., \( m \)), knowing that the process was even further (at some point \( x \) at distance at least \( 2m \)). In Inequality (38) we bound this by the probability of staying within distance \( m \).

By the first item of Claim 25, the r.h.s of Inequality (38) is at least some positive constant (again, independent of \( m \)). Overall, for any \( m \geq 2 \), we have:

\[
\Pr(\|Z(m)\| \geq m) \geq \Pr(\|Z(m)\| \geq m \mid A_{\text{far}}) \cdot \Pr(A_{\text{far}}) \geq \gamma,
\]

for some constant \( \gamma > 0 \) (independent of \( m \)). Next, using the second item of Claim 25 with \( \delta = 1 - \frac{\gamma}{2} \), we get that there exists a large enough constant \( c' > 0 \) (again, independent of \( m \)), such that:

\[
\Pr(\|Z(m)\| \leq c'm) \geq \delta.
\]

(39)

Hence, using a union bound argument, we have:

\[
\Pr(m \leq \|Z(m)\| \leq c'm) \geq \Pr(\|Z(m)\| \geq m) + \Pr(\|Z(m)\| \leq c'm) - 1
\]

\[
\geq \gamma + \delta - 1 = \frac{\gamma}{2} > 0.
\]

This establishes Eq. (35) and thus concludes the proof of Lemma 20.
C.2.3 Proof of Lemma 21 (upper bound)

This section is dedicated to the proof of Lemma 21:

Lemma 21 (re-stated). For any constant \( \alpha > 0 \), there exists a constant \( c' > 0 \) such that, for any integer \( m \in [2, \alpha \ell_{\text{max}}] \) and any \( x \in \mathbb{R}^2 \), we have

\[
\Pr(Z(m) \in B(x)) \leq \frac{c' \log^2 m}{m^2}.
\]

Proof. Let \( \alpha > 0 \) and \( m \in [2, \alpha \ell_{\text{max}}] \). Due to the monotonicity property stated in Corollary 5, it is sufficient to prove this result for \( x = 0 \). Indeed, for any \( x \in \mathbb{R}^2 \), the sets \( B(0) \setminus B(x) \) and \( B(x) \setminus B(0) \) have the same area \( A \), and

\[
\Pr(Z(m) \in B(x) \setminus B(0)) \leq A \max_{y \in B(x) \setminus B(0)} \{ p_{\|Z(m)\|}(y) \}
\]

\[
\leq A \min_{y \in B(0) \setminus B(x)} |\{ p_{\|Z(m)\|}(y) \}|
\]

\[
\leq \Pr(Z(m) \in B(0) \setminus B(x)),
\]

where the second inequality is due to the monotonicity property and the fact that any point in \( B(x) \setminus B(0) \) is at distance more than 1 from the origin, and hence, further from 0 than any point in \( B(0) \setminus B(x) \). This shows that \( \Pr(Z(m) \in B(x)) \leq \Pr(Z(m) \in B(0)) \), hence it is sufficient to prove the required upper bound for \( x = 0 \).

Intuitively, to establish this, we say that with high probability, there is some step \( s \leq m \) for which \( Z(s) \) is “distant” (at least \( cm/\log m \)). Conditioning on this, the probability to be located in \( B(0) \) at step \( m \) is found out to be small, due to the monotonicity of the process (Corollary 5). Formally, consider a (small) positive constant \( c \), and let \( \mathcal{A} \) be the event that there is some \( s \leq m \) for which \( \|Z(s)\| \geq cm/\log m \).

Consider \( B(0) \) the ball of radius 1 with center 0. Write

\[
\Pr(Z(m) \in B(0)) = \Pr(Z(m) \in B(0) \cap \mathcal{A}) + \Pr(Z(m) \in B(0) \cap \neg \mathcal{A})
\]

\[
\leq \Pr(Z(m) \in B(0) \mid \mathcal{A}) + \Pr(\neg \mathcal{A}),
\]

By the first item of Claim 24, taking \( c \) to be sufficiently small, we have

\[
\Pr(\neg \mathcal{A}) = O(m^{-2}).
\]

In order to express the remaining term of Eq. (40), we will denote in the following equation \( Z^x \) the Cauchy process on \( \mathbb{R}^2 \) with cut off \( \ell_{\text{max}} \) starting with \( Z(0) = x \). Since our process was defined to start at 0, we have \( Z = Z^0 \). Remark that the law of \( Z^x \) is obtained by a translation of that of \( Z^0 \).
With this notation in mind, we have, using the Markov property for the second inequality:

\[ \Pr(Z^0(m) \in B(0) \mid A) \leq \max_{s \leq m} \Pr(Z^0(m) \in B(0) \mid \|Z^0(s)\| \geq cm/\log m) \]

\[ \leq \max_{s \leq m} \sup_{\|x\| \geq cm/\log m} \Pr(Z^x(m - s) \in B(0)) \]

\[ = \max_{s \leq m} \sup_{\|x\| \geq cm/\log m} \Pr(Z^x(s) \in B(0)) \]

\[ = \max_{s \leq m} \sup_{\|x\| \geq cm/\log m} \Pr(Z^0(s) \in B(-x)) \]

\[ = \max_{s \leq m} \sup_{\|x\| \geq cm/\log m} \Pr(Z(s) \in B(x)) \]

Use now Corollary 5 that gives \( p^{Z(m)}(x) \leq \frac{1}{\pi \|x\|^2} \). Hence, for any \( x \in \mathbb{R}^2 \) with \( \|x\| > 1 \), we have

\[ \Pr(Z(m) \in B(x)) = \int_{B(x)} p^{Z(m)}(y) dy \leq \int_{B(x)} \frac{1}{\pi (\|x\| - 1)^2} dy = \frac{1}{(\|x\| - 1)^2}. \]

Let \( m(c) \) be the largest integer \( m > 0 \) such that \( cm/\log m \leq 2 \). For \( m > m(c) \), we have

\[ \Pr(Z(s) \in B(x)) \leq \max_{s \leq m} \frac{1}{(cm \log m - 1)^2} = \frac{1}{(cm \log m - 1)^2} \]

Overall, we find that, for \( m > m(c) \)

\[ \Pr(Z(m) \in B(0)) \leq \frac{1}{(cm \log m - 1)^2} + \frac{c'}{m^2}, \]

which we can bound by \( \frac{c2 \log^2 m}{m^2} \) for some constant \( c_2 > 0 \). Since \( m(c) \) is a constant, there is some other constant \( c_3 > 0 \) for which, for any \( m \in [2, m(c)] \), we have \( \Pr(Z(m) \in B(0)) \leq \frac{c_3 \log^2 m}{m^2} \). We then obtain, for any \( m \geq 2 \),

\[ \Pr(Z(m) \in B(0)) \leq \frac{\max\{c_2, c_3\} \log^2 m}{m^2}, \]

which concludes the proof of Lemma 21.
REFERENCES AND NOTES


